ELEMENTARY
MATHEMATICAL ANALYSIS

VOLUME II

ELEMENTARY MATHEMATICAL ANALYSIS

BY

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VOLUME II

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PREFACE

The second volume of a freshman text in mathematics must continue and extend certain topics begun in the first volume and it should not omit any additional topics which are to be used later in the study of the calculus. These two objectives have been always before the author in the preparation of the pages that follow.

The first chapter, The Number System of Elementary Mathematics, logically belongs at the beginning of the first volume, but, in the experience of many teachers, there is an advantage in postponing the discussion until the students have attained a more mature mathematical outlook.

The *theory of poles and polars* as here presented, though brief, contains the notions which are fundamental. When a more complete theory is given, many instructors feel obliged to omit it for lack of time.

The sections on *determinants* are adequate as preparation for solid analytical geometry, for a more advanced course in the theory of equations, and for projective transformations in space of two, three, or n dimensions. Their introduction here, however, has an immediate justification not only in the customary applications to the solution of linear non-homogeneous equations, but also in the apt applications to analytical geometry which make a strong appeal to the student's imagination (see pages 52 to 55).

To a casual observer the treatment of *translation of axes* may appear inadequate, but, as a matter of fact, beginning with the first chapter of Volume I, the *process* of translation has been applied to every type of equation and function to which it is properly applicable; hence there remains only to make in Chapter IX a precise statement of the technical terms associated with this process.

In the treatment of *partial fractions* the customary division into Cases I, II, III, and IV has been departed from; also, the problems selected are reasonably simple as to the algebra involved in their solution. Attention has been given to the frequently neglected case (needed for integration) of the rational fraction whose denominator is a power of a single linear or irreducible quadratic expression.

Familiarity with the notions of *parameter* and *pencil* is acquired by continual and persistent observation of the effect on a curve that is obtained by varying one or more constants in each equation.

Volumes I and II afford complete preparation for the formal course in calculus usually given in the sophomore year.

The author wishes again to express her indebtedness to Professors G. A. Bliss and H. E. Slaught for their counsel and to Professor Lincoln La Paz and Mr. Nelson Dunford for reading and criticizing much of the manuscript.

MAYME IRWIN LOGSDON.

THE UNIVERSITY OF CHICAGO,
December, 1932

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ELEMENTARY MATHEMATICAL ANALYSIS

CHAPTER I

THE NUMBER SYSTEM OF ELEMENTARY MATHEMATICS

Some of the most fascinating chapters in the history of elementary mathematics are those which trace the origin and growth of number concepts. The oldest mathematical document extant is an Egyptian papyrus called the *Rhind papyrus,** now in the British Museum. It was written by Ahmes some time before 1700 B.C. and was founded on an earlier work which probably dated back as far as 3400 B.C. The clumsy notations used by Ahmes and by all other primitive peoples before the introduction of the present-day *Arabic notation* by the Hindus in the eighth century prevented the making of rapid progress in mathematical manipulation. It is not surprising that centuries elapsed with little growth of mathematical notions.†

In this chapter is given a very brief sketch of the principal steps in the growth of the number system of elementary mathematics. For a more complete discussion the student is referred to standard texts on the history of mathematics.

1. The real number system. In the first stages of the development of the science of mathematics as well as in the mathematical development of the individual, the only kind of number known (and needed) is the set of **positive integers.** For a subtrahend which is less than the minuend and for a divisor which is a factor of the dividend the result of performing one or more of the elementary operations of arithmetic, *viz.*, addition, subtraction, multiplication, and division, on numbers of the set will give a number of the set; *e.g.*, $7 + 5 = 12$, $10 - 3 = 7$, $6 \times 5 = 30$, $44 \div 11 = 4$.

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* A facsimile of this papyrus with translation and commentary was published in 1929 by the Mathematical Association of America and is for sale by the Open Court Co., Chicago.

† See "The Evolution of Numbers," a play by Prof. H. E. Slaught, which appeared in the *Mathematics Teacher*, Vol. XXI, October, 1928.

There comes a time, however, when **negative integers** and **zero** are introduced into the set so that the subtrahend may not be restricted and when **rational fractions** are introduced into the set (a rational fraction is the quotient of two integers) so that the divisor may not be restricted; *e.g.*, $14 - 25 = -11$, $15 \div 2 = 7\frac{1}{2}$, $5 \div 20 = \frac{1}{4}$.

At this stage the number system consists of the following elements and operations:

NUMBER SYSTEM—(A)

Elements (real)	Operations
Rational numbers	Addition
Integers	
Positive	Subtraction
Negative	
Zero	
Rational fractions	Multiplication
Positive	
Negative	Division

Division by zero is not permitted. With this restriction the four operations on elements of the system give elements of the system.

2. Irrational numbers. The next extension of the number system is motivated by the study of powers and roots of numbers. These are computed by operations called, respectively, **involution** and **evolution.** The result of raising any number of the system (A) to a power p, where p is a positive integer, is again a number of (A). The same cannot be said of the result of extracting the pth root of any number of (A). For example, 2 is a square root of 4 and 2 is a cube root of 8 but there is no number in (A) which when squared gives 2,[*] or 3, or 5. However, an approximate value of, say, $\sqrt{2}$ can be found, *i.e., a rational number whose square*

---

[*] *Proof that no rational number is the square root of* 2. Assume that there is a rational number which is the square root of 2 and denote it by p/q where p and q have no common factor. Then $p^2/q^2 = 2$ and $p^2 = 2q^2$. Now p must be an even number, since there is a factor 2 on the right. Consequently q must be an odd number. Denote p by $2s$. Then $(2s)^2 = 2q^2$, and $q^2 = 2s^2$. The last equality cannot hold since q is an odd number; hence the first assumption is untenable. *q.e.d.*

differs from 2 by as little as we please. Indeed two such approximations can be obtained such that the square of the one is less than 2 while the square of the other is greater than 2. They are approximations to the roots of the pure quadratic equation

$$x^2 - 2 = 0.$$

The following equalities may be verified:

$$\begin{array}{llll}
(1.4)^2 & = 1.96, & (1.5)^2 & = 2.25, \\
(1.41)^2 & = 1.9881, & (1.42)^2 & = 2.0164, \\
(1.414)^2 & = 1.999393, & (1.415)^2 & = 2.002225, \\
(1.4142)^2 & = 1.99996164; & (1.4143)^2 & = 2.00024449.
\end{array}$$

The square of 1.4142 (and of -1.4142) differs from 2 by 0.00003836, the square of 1.4143 differs from 2 by 0.000244449. We say that we have found approximations to the square root of 2 which are correct to the fourth and to the third decimal places, respectively. Another way of expressing the fact that these numbers are approximations to the square root of 2 is

$$1.4142 < \sqrt{2} < 1.4143,$$

though the latter statement does not indicate which of the two approximations is the better.

When a closer approximation is desired it can be obtained. In fact

$$1.4142135 < \sqrt{2} < 1.4142136$$

and the square of 1.4142135 differs from 2 by less than 0.0000002. (A decimal fraction is a rational number; *e.g.*, $1.4142 = {}^{14142}\!/_{10000}$.)

Again, the equation $x^5 + x^4 + x^3 + x^2 + x + 2 = 0$ has no rational root, nor is there any number expressible by radicals which is a solution of it. However, an approximation to a root is -1.215 and it can be verified that when -1.215 is substituted for x in the equation under discussion the left member differs from zero by less than 0.001. When a closer approximation is desired it can be obtained by Horner's method which is explained in the next chapter. (See Sec. 15.)

Real numbers which are not rational are called **irrational numbers.** In particular, an irrational number which is the root of a rational integral equation with rational coefficients is an **irrational algebraic number.** *The value of an irrational number may be approximated in decimals to any desired degree of accuracy.*

Note 1. There is no number in (A) which approximates a square root of -2; *i.e.*, there is no rational number which is approximately a solution of the equation $x^2 + 2 = 0$, though an approximation to a solution of $x^3 + 2 = 0$ is easily found.

Note 2. The number π whose approximate value is 3.14159265 is not an irrational algebraic number. It is shown in higher mathematics that π cannot be the root of a rational integral equation with rational coefficients. It is called a **transcendental** (irrational) **number.**

We agree to use as our number system all irrational numbers and all rational numbers, *i.e.*, the set of all **real numbers.** This extended number system and the operations which again give numbers of the system are exhibited in the following table:

Real Number System—(B)

Elements	Operations
	Addition
Rational numbers	Subtraction
	Multiplication
	Division
Irrational numbers	Involution
	Evolution

Division by zero is excluded and the operation of evolution is considerably restricted. For example, *an indicated even root of a negative number does not belong to the set.*

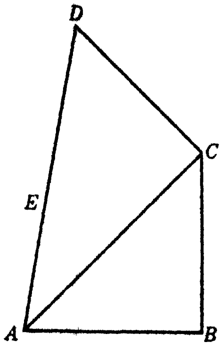

Fig. 1.

Geometrical construction of line segments whose lengths are expressed by indicated square roots. While irrational numbers can be only approximated algebraically, it is sometimes a very simple matter to construct line segments whose lengths are correctly represented by irrational numbers. For example, in Fig. 1 let the length AB be one unit. Construct $BC = AB$ and perpendicular to AB; construct $DC = AB$ and perpendicular to AC, and construct $DE = AB$. Then

$$AC = \sqrt{2},$$
$$AD = \sqrt{3},$$
$$AE = \sqrt{3} - 1.$$

3. The complex number system. The restrictions on the operation of *evolution* in the number system (B) lead to the next extension, in fact the extension which motivated this chapter. We include in the new number system, denoted by (C), *indicated roots which can not be approximated by rational numbers, e.g.*, $\sqrt{-2}$, $\sqrt[4]{-7}$, etc. We also, by means of the definitions of fractional exponents and of negative exponents,* remove the restriction that p, the index of the root, shall be a positive integer. The new number is called a **complex or imaginary number** to distinguish it from real numbers. The new number system is called the **complex number system.** It includes, as a part, system (A) and also system (B).

Complex Number System—(C)

Elements	Operations
Real numbers	Addition
Rational numbers	Subtraction
Irrational numbers	Multiplication
Imaginary numbers	Division
	Involution
	Evolution

Apart from the restriction that division by zero is not permitted, the result of performing a finite number of the operations of the complex number system on numbers of the system is a complex number.

The number system in (C) is now adequate for the mathematics of everyday life, for the mathematics of statistics, navigation, astronomy, life insurance, accounting, banking, the calculus and its applications, and many phases of engineering. There have come times, however, in the development of the science of mathematics and of its applications, when it has been desirable to introduce other kinds of elements into the number system. The result has been the development of profound and scholarly mathematical doctrine as well as many important applications, as, for example, the applications in physics and chemistry which have given to us the radio, television, etc.

4. Arithmetical operations on complex numbers. A complex number may be written as the sum of a real part and an imaginary part, as $5 + \sqrt{-7}$, $3 - \sqrt{-2}$, $a + \sqrt{-b}$, where b is a positive number. In writing complex numbers it is convenient to use a symbol i to represent the indicated *positive* square root of -1, *viz.*,

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* Logsdon, "Elementary Mathematical Analysis," Vol. I, Chap. V.

$$i = \sqrt{-1}.$$

This symbol has the property that $i^2 = -1$, $i^3 = -i$, $i^4 = +1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$, etc.

The square root of any negative number may be represented as the product of a real number and i, called an **imaginary unit.** (The **real** units are ± 1.)

Thus $\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i$, $\sqrt{-5} = \sqrt{-1}\sqrt{5} = i\sqrt{5}$, etc. The complex numbers $4 + \sqrt{-1}$, $5 - \sqrt{-7}$, $-2 - \sqrt{-3}$ are written $4 + i$, $5 - i\sqrt{7}$, $-2 - i\sqrt{3}$, respectively; in fact, any complex number may be written in the form

$$a + bi$$

where a and b are real (though not necessarily rational), a is the real part and bi is the imaginary part. The complex numbers $a + bi$ and $a - bi$ are called **conjugate** complex numbers. The real numbers a and b are called **components** of the complex number $a + bi$ while the components of the conjugate complex number are a and $-b$.

A **pure imaginary number** is one for which the first component is zero, *e.g.*, $7i$, $-3i$, $i\sqrt{5}$. When the second component, the coefficient of i, is zero, the number is a real number, *e.g.*, 4, -10, $\frac{1}{2}\sqrt{11}$. Thus the set of all complex numbers is seen to include all real numbers as a part, as was stated above.

DEFINITION. *Two complex numbers will be considered equal if, and only if, the real parts are equal and the imaginary parts are equal.* Thus $a + bi = c + di$ implies the two separate equations

$$a = c,$$
$$b = d.$$

Cor. The complex number $a + bi$ is equal to the (complex) number zero if, and only if, $a = 0$ and $b = 0$.

We wish to perform on complex numbers the operations of (C). The following may be taken as definitions:

I. Addition. The sum of $a + bi$ and $c + di$ is $(a + c) + (b + d)i$.

II. Subtraction. The result of subtracting $c + di$ from $a + bi$ is $(a - c) + (b - d)i$. This is equal to the sum of $a + bi$ and $-c - di$.

III. Multiplication. The product of $a + bi$ by $c + di$ is $(ac - bd) + (ad + bc)i$. It is obtained by ordinary algebraic multiplication. When i^2 occurs it is replaced by -1.

$$\begin{array}{l} a + bi \\ \underline{c + di} \\ ac + bci \\ \underline{\quad + adi - bd} \\ ac + (bc + ad)i - bd \end{array}$$

Cor. The product of $a + bi$ by its conjugate $a - bi$ is the real number $a^2 + b^2$.

IV. Division. The quotient of $a + bi$ by $c + di$ can be expressed as a complex number in the following manner:

$$\begin{aligned} \frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i. \end{aligned}$$

V. Extraction of roots. In this paragraph we shall discuss an *algebraic* method of finding the square roots of a complex number. In Section 7, after a geometrical representation of complex numbers has been agreed upon, the problem of extracting roots of order 2 or more can be solved geometrically in a simpler manner than is possible with an algebraic solution.

It is desired to find the complex number, $x + yi$, x and y real numbers, such that

$$(x + yi)^2 = a + bi.$$

Now $(x + yi)^2 = x^2 - y^2 + 2xyi$. This can, by the definition on page 6, be equal to the given number $a + bi$ if, and only if, the two real numbers x and y satisfy the two simultaneous quadratic equations,

$$\begin{aligned} x^2 - y^2 &= a, \\ 2xy &= b. \end{aligned}$$

There will always be real solutions of this system of equations, though the solutions are not necessarily rational.

ILLUSTRATIVE EXAMPLE. Find the square root of $5 + 12i$.

SOLUTION. We seek real solutions of the two equations

$$\begin{aligned} x^2 - y^2 &= 5, \\ xy &= 6. \end{aligned}$$

When $y = 6/x$ is substituted in the first equation and the resulting equation cleared of fractions, we have

$$x^4 - 5x^2 - 36 = 0,$$
$$(x^2 - 9)(x^2 + 4) = 0.$$

The only real solutions are $x = 3$ and $x = -3$. The corresponding values of y are $y = 2$ and $y = -2$. Hence, the two square roots of $5 + 12i$ are $x + yi = 3 + 2i$ and $-3 - 2i$.

EXERCISE I

1. Find the sum of $3 - 5i$ and $-4 + 7i$; of $4 + i$ and $4 - i$; of $2i$ and $7 + 8i$; of $2 + 4i$ and $-2 - 4i$; of $c + di$ and $c - di$; of $c + di$ and $-c - di$.

2. Find the product of $3 + 2i$ and $-1 - 2i$; of $2i$ and $7i$; of $8 - 3i$ and $2 + 5i$; of $x + yi$ and $-x - yi$; of $c + di$ and $c - di$; of $x + yi$ and i.

3. Express as complex numbers the following quotients:

$$\frac{3 - i}{2 + i};\quad \frac{7 - 5i}{i};\quad \frac{2 + 3i}{7 + 5i};\quad \frac{7}{2 - 5i};\quad \frac{a + bi}{a - bi};\quad \frac{x + yi}{-x - yi};\quad \frac{3 - 6i}{1 + i}.$$

4. Find the square roots of the following:

$$3 - 4i;\quad 3 + 4i;\quad 8 + 15i;\quad -35 - 12i;\quad 3 - 2i;\quad 4 + 5i.$$

5. Construct a line segment whose length is $\sqrt{58}$; $\sqrt{10}$; $\sqrt{7}$.

5. The Argand diagram. Geometrical representation of complex numbers. Before discussing a method of adding, multiplying, etc., two complex numbers geometrically, let us recall methods learned in elementary geometry of performing geometrically the operations of arithmetic on real numbers. Let the real numbers a and b be represented by line segments whose lengths are a units and b units. We can then *construct with ruler and compasses* the line segments which represent the real numbers $a + b$, $a - b$, ab, a/b, a^2, $\sqrt{ab}$. These constructions are shown in Fig. 2.

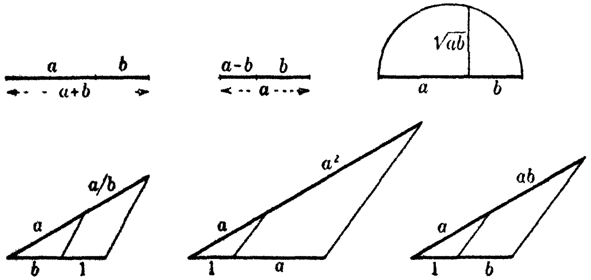

FIG. 2.—Operations on real numbers performed geometrically.

A very useful method of linking the arithmetic of complex numbers, which was explained in the last section, with geometrical notions and constructions is *to associate the complex number* $x + yi$ *with the point* $P(x, y)$ *in the* (x, y) *plane* in which a rectangular system of coordinates has been established. This sets up a one-to-one correspondence between the system of all complex numbers and the points of the plane. Moreover, all numbers $x + yi$ for which $y = 0$ will correspond to points on the x-axis, which will be called the **axis of reals**, and all numbers $x + yi$ for which $x = 0$ will correspond to points on the y-axis, called the **axis of pure imaginaries.** The drawing in Fig. 3 shows the points which are associated with certain complex numbers according to this agreement. It will be noted that two numbers which are conjugate are associated with points which are symmetric with respect to the X-axis, and that *the distance OP from the origin to the point P which is associated with the complex number* $x + yi$ *is* $\sqrt{x^2 + y^2}$. This real positive number is called the **modulus, or absolute value,*** of the complex number $x + yi$. It is sometimes written $|x + yi|$ and is frequently designated by the Greek letter ρ (rho). The line segment itself drawn from O to the point P is called the **radius vector** of the point.

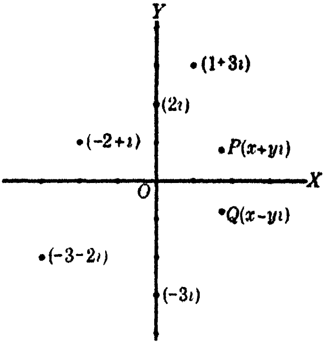

FIG. 3.—Complex numbers and associated points.

Note. Two conjugate complex numbers have equal moduli; i.e.,

$$|x + yi| = |x - yi| = \sqrt{x^2 + y^2}.$$

We can associate with a complex number $x + yi$ an angle θ through which (positive) OX must turn in order that it may coincide with OP. The value of the angle θ is not unique. For example, in Fig. 4, the value of θ may be taken as 30°, −330°, 390°, in fact $30° + n.360°$ where n is any integer. For each of these values of θ we have

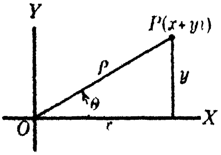

FIG. 4.—$x + yi = \rho(\cos\theta + i\sin\theta)$.

$$\tan\theta = \frac{y}{x};$$

hence we can choose any convenient one of the above values and

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* Logsdon, "Elementary Mathematical Analysis," Vol. I, p. 24.

call it the amplitude or argument of the complex number $x + yi$; for example, denoting amplitude by the symbol "*am*" we write $am(1 + i) = 45°$, $am(-3) = 180°$, $am(-6i) = -90°$.

For any complex number $x + yi$, Fig. 4 shows that the following relations exist between the pair of numbers (x, y) and the pair (ρ, θ).

$$\begin{aligned} x &= \rho \cos \theta, \\ y &= \rho \sin \theta. \end{aligned} \tag{1}$$

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2}, \\ \theta &= \arc\tan \frac{y}{x}. \end{aligned} \tag{2}$$

When a number is given in the form $x + yi$, equations (2) enable us to compute ρ and θ and we write

$$x + yi = \rho(\cos \theta + i \sin \theta). \tag{3}$$

Similarly, when a complex number is given in the form $\rho(\cos \theta + i \sin \theta)$, called the **trigonometric form** and also the **polar form** of the complex number, equations (1) enable us to compute the x and y components so that we can write the number in its Cartesian form, $x + yi$.

For brevity, we often speak of the complex number P when we mean the point P associated with the complex number; also, we say the point $3 + 2i$, the point $1 - 5i$, etc., when we mean the point which is associated with $3 + 2i$, with $1 - 5i$, etc.

ILLUSTRATIVE EXAMPLE 1. Write the trigonometric form of the complex number $2 + 2i\sqrt{3}$.

SOLUTION. From equations (2), we get $\rho = \sqrt{x^2 + y^2} = \sqrt{2^2 + 2^2 \cdot 3} = 4$; from equations (1), we get $\cos \theta = x/\rho = 2/4 = 1/2$, $\sin \theta = y/\rho = 2\sqrt{3}/4 = \frac{1}{2}\sqrt{3}$. Hence, $\theta = 60°$ and we write

$$2 + 2i\sqrt{3} = 4(\cos 60° + i \sin 60°).$$

ILLUSTRATIVE EXAMPLE 2. Write the trigonometric form of the complex number $-3i$.

SOLUTION. From (2), $\rho = \sqrt{0^2 + (-3)^2} = 3$; from (1), $\cos \theta = 0$, $\sin \theta = -3/3 = -1$; hence $\theta = -90°$ or $270°$, and we write

$$-3i = 3(\cos 270° + i \sin 270°) \text{ or } 3[\cos(-90°) + i \sin(-90°)].$$

ILLUSTRATIVE EXAMPLE 3. Write the Cartesian form of the complex number whose trigonometric form is $6(\cos 225° + i \sin 225°)$.

SOLUTION. We have given $\rho = 6$, $\theta = 225°$. From (1), $x = 6 \cos 225° = -3\sqrt{2}$, $y = 6 \sin 225° = -3\sqrt{2}$. Hence

$$6(\cos 225° + i \sin 225°) = -3\sqrt{2} - 3i\sqrt{2}.$$

EXERCISE II

1. Plot the points in the Argand plane which are associated with the following complex numbers:

$$2 + 5i,\quad 3 - i,\quad -1 + 4i,\quad -1 - 4i,\quad -3,\quad 2i,\quad 7 - 3i,\quad 4$$

2. Find the moduli of the numbers in 1.

3. Find the sines and cosines of the amplitudes of the numbers in 1.

4. Plot the points which are associated with the conjugates of the numbers in 1.

5. Express in trigonometric form:

$$1 + i,\quad 2 + 2i,\quad -\frac{1}{2} + \frac{i}{2}\sqrt{3},\quad -\frac{1}{2} - \frac{i}{2}\sqrt{3},\quad 3,\quad -3,\quad 3i,\quad -3i.$$

6. Write in Cartesian form the complex numbers:

(*a*) $2(\cos 30^\circ + i \sin 30^\circ)$; (*b*) $\frac{1}{3}(\cos 210^\circ + i \sin 210^\circ)$;
(*c*) $5(\cos 90^\circ + i \sin 90^\circ)$; (*d*) $3(\cos 150^\circ + i \sin 150^\circ)$;
(*e*) $2(\cos 180^\circ + i \sin 180^\circ)$; (*f*) $4(\cos 360^\circ + i \sin 360^\circ)$.

7. Plot the points associated with the complex numbers in Problem 6.

6. The operations on complex numbers performed geometrically. I. Addition. To find the point which is associated with the complex number

$$x + yi = (a + bi) + (c + di),$$

we complete the parallelogram which has for sides the two radii vectores OA and OC. (Fig. 5.) The abscissa of P in the figure is $a + c$ and its ordinate is $b + d$. Hence from the definition of the sum of two complex numbers, P is the desired point. The student should verify that the construction is correct for complex numbers when their associated points are not both in the first quadrant.

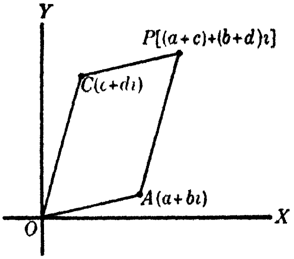

Fig. 5—Addition of complex numbers.

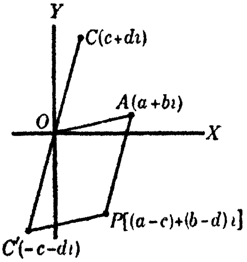

Fig. 6.—Subtraction of complex numbers

II. Subtraction. To subtract $c + di$ from $a + bi$ geometrically, we first locate the point C' which is associated with the number $-c - di$, then we construct the sum of $a + bi$ and $-c - di$. (See Fig. 6.)

Note. The points $C(c + di)$ and $C'(-c - di)$ are symmetric with respect to the origin.

EXERCISE III

1. *Add geometrically and arithmetically:*

(a) $-2 + 3i$ and $1 - 4i$;
(b) $2(\cos 30° + i \sin 30°)$ and $5(\cos 150° + i \sin 150°)$;
(c) $5 - 2i$ and $-5 + 3i$;
(d) $(\cos 90° + i \sin 90°)$ and $(\cos 270° + i \sin 270°)$;
(e) $2 - i$ and $2 + i$;
(f) $3(\cos 180° + i \sin 180°)$ and $7(\cos 360° + i \sin 360°)$.

2. Subtract, as indicated, geometrically and arithmetically:

(a) $(-2 + 3i) - (5 + i)$;
(b) $4(\cos 135° + i \sin 135°) - 3(\cos 45° + i \sin 45°)$;
(c) $(7 - i) - (4)$;
(d) $2(\cos 240° + i \sin 240°) - 4(\cos 120° + i \sin 120°)$.

III. Multiplication. When two complex numbers are to be multiplied geometrically it is convenient to put them into trigonometric form before performing the multiplication. The theorem for multiplication is stated thus: *The modulus of the product of two complex numbers is the* **product** *of their moduli; the amplitude of the product is the* **sum** *of their amplitudes.*

Proof. Let the two given complex numbers be $\rho_1(\cos \theta_1 + i \sin \theta_1)$ and $\rho_2(\cos \theta_2 + i \sin \theta_2)$. Their product will be

$$\rho_1\rho_2(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2).$$

To express this as a complex number, write $(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) = \cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)$. Hence the required product, *viz.*,

$$\rho_1\rho_2[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)],$$

is a complex number in trigonometric form whose modulus is $\rho_1\rho_2$ and whose amplitude is $\theta_1 + \theta_2$. *q.e.d.*

Cor. 1. The amplitude is unchanged when a given complex number is multiplied by a real number.

Cor. 2. The modulus is unchanged when a given complex number is multiplied by a number whose modulus is unity.

There are many complex numbers with unit modulus. They are the numbers which are associated with the points on the circle whose equation is $x^2 + y^2 = 1$, *viz.*, the **unit circle.** Each of these numbers is called a **complex unit** and is expressible in trigonometric form by $\cos \alpha + i \sin \alpha$ where $0 \leqq \alpha < 360°$.

Cor. 3. Multiplication by i rotates the radius vector through $+90°$. (Fig. 7.)

Cor. 4. The modulus of the square of $\rho(\cos\theta + i\sin\theta)$ is ρ^2; the amplitude is 2θ.

IV. Division. The theorem for division is stated thus: *The modulus of the quotient of two complex numbers is the modulus of the dividend* **divided by** *the modulus of the divisor; the amplitude of the quotient is the amplitude of the dividend* **minus** *the amplitude of the divisor.*

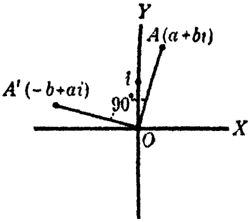

FIG. 7.—The product of a complex number by i.

Proof. The quotient of $\rho_1(\cos\theta_1 + i\sin\theta_1)$ by $\rho_2(\cos\theta_2 + i\sin\theta_2)$ is

$$\frac{\rho_1(\cos\theta_1 + i\sin\theta_1)}{\rho_2(\cos\theta_2 + i\sin\theta_2)} = \frac{\rho_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{\rho_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{\rho_1}{\rho_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]. \qquad q.e.d.$$

ILLUSTRATIVE EXAMPLE. Construct geometrically the point P which is associated with $x + yi = (-\tfrac{1}{2} + i)(1 + 3i)$. Compute the values of x and y.

CONSTRUCTION. In Fig. 8 the points A and B are associated with $1 + 3i$ and $-\tfrac{1}{2} + i$, respectively. The point P must satisfy two conditions; *viz.*, the angle XOP = angle XOA + angle XOB and length OP = length $OA \cdot$ length OB. The length OP may be constructed as in Fig. 2, Sec. 5, and the angle may also be constructed with ruler and compasses.

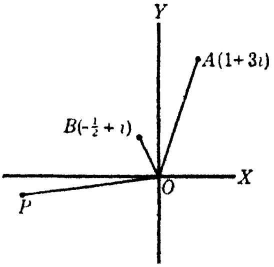

FIG. 8.—The product of two complex numbers.

SOLUTION. Am $(1 + 3i) = \text{arc}\tan 3 = 71° 57$, am $(-\tfrac{1}{2} + i) = \text{arc}\tan(-2) = 116°.55$; hence am $(x + yi) = 71°.57 + 116°.55 = 188°.12$. Also $|1 + 3i| = \sqrt{10}$, $|-\tfrac{1}{2} + i| = \tfrac{1}{2}\sqrt{5}$, therefore $|x + yi| = \sqrt{10} \cdot \tfrac{1}{2}\sqrt{5} = \tfrac{5}{2}\sqrt{2}$.

In Cartesian form $x + yi = -\tfrac{7}{2} - \tfrac{1}{2}i$.

EXERCISE IV

1. Restate corollaries 1, 2, 3, and 4 under multiplication so that they will apply to division.

2. Construct geometrically the points associated with the products indicated:

(*a*) $(3 + 5i)i$;
(*b*) $(3 + 5i)(-1)$;
(*c*) $(2 - i)(-1 + i)$;
(*d*) $(2i)(i)$;
(*e*) $(2 + 3i)(2 - 3i)$;
(*f*) $(-1 - i)(-1 + i)$.

3. Put the products in Problem 2 into trigonometric and into Cartesian form.

4. Construct geometrically the points associated with the quotients indicated:

(a) $\frac{2+i}{i}$; (b) $\frac{5-3i}{3+5i}$; (c) $\frac{3i}{1-i}$;

(d) $\frac{-3}{2+i}$; (e) $\frac{7+3i}{2}$; (f) $\frac{4-3i}{1+2i}$.

5. Express the quotients in Problem 4 in trigonometric and in Cartesian form.

7. De Moivre's theorem. The law for multiplication of complex numbers gives

$$(\cos \alpha + i \sin \alpha)^2 = \cos 2\alpha + i \sin 2\alpha.$$

If each member of this equation is multiplied by $\cos \alpha + i \sin \alpha$, the result is

$$(\cos \alpha + i \sin \alpha)^3 = \cos 3\alpha + i \sin 3\alpha.$$

It can be proved that the equation

$$(\cos \alpha + i \sin \alpha)^n = \cos n\alpha + i \sin n\alpha,$$

is true for *any real value of n*. This theorem is a special form of De Moivre's* theorem. It will not be proved here.

The complex number $\cos \alpha + i \sin \alpha$ is associated with a point on the unit circle for every value of α and any complex number which is associated with a given point on the unit circle will have the same form. The points associated with $\cos 2\alpha + i \sin 2\alpha$, $\cos n\alpha + i \sin n\alpha$, etc., are also on the unit circle. Hence from De Moivre's theorem we conclude that the square, the cube, the fourth power, etc., of any complex number whose modulus is unity is itself a number whose modulus is unity and consequently lies on the unit circle. The same is true of square roots, cube roots, fourth roots, etc., of the complex units.

We shall now see how the above theorem of De Moivre enables us to find the two square roots, the three cube roots, . . . the n nth roots of a number whose modulus is unity.

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* The theorem is due to Abraham De Moivre (1667–1754) who was born in France but lived in England after the age of seventeen. A proof of the theorem is given in Palmer and Leigh, "Plane and Spherical Trigonometry," pp. 172*ff*., McGraw-Hill Book Company, Inc , New York. The student will be able to prove it after he has studied Chap. XIII in this book. See Problem 3, Exercise LV.

ILLUSTRATIVE EXAMPLE 1. Find the two square roots of $\cos \Theta + i \sin \Theta$.

SOLUTION. We seek all values of θ which satisfy the equation

$$(\cos \theta + i \sin \theta)^2 = \cos \Theta + i \sin \Theta;$$

i.e.,

$$\cos 2\theta + i \sin 2\theta = \cos \Theta + i \sin \Theta.$$

In view of the definition of equality (page 6) the last relation subsists if, and only if,

$$\sin 2\theta = \sin \Theta, \qquad \cos 2\theta = \cos \Theta.$$

Hence, $2\theta = \Theta + 2k\pi$ where k is any integer, positive, negative, or zero. Let $k = 0$ and 1 and call θ_0 and θ_1 the corresponding values of the angle; then $\theta_0 = \frac{1}{2}\Theta$ and $\theta_1 = \frac{1}{2}\Theta + \pi$. The two angles θ_0 and θ_1 when in standard position have distinct terminal sides. Let k take an integral value not 0 or 1. The corresponding value of θ will have its terminal side coincident with that of θ_0 if k is an even number and with θ_1 if k is an odd number. Hence there are only two square roots of $\cos \Theta + i \sin \Theta$ and their amplitudes may be written $\frac{1}{2}\Theta$ and $\frac{1}{2}\Theta + \pi$.

To construct the two square roots of $\cos \Theta + i \sin \Theta$, draw the unit circle (Fig. 9) and locate the point A whose amplitude is Θ. Construct $\theta_0 = \frac{1}{2}\Theta$, thus locating the point P_0. The point P_1 is diametrally opposite the point P_0.

The points P_0 and P_1 are associated with the complex numbers which are square roots of $\cos \Theta + i \sin \Theta$.

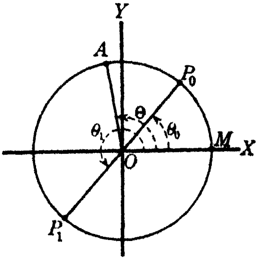

FIG. 9.—The two square roots of a complex number.

ILLUSTRATIVE EXAMPLE 2. Determine the cube roots of the number associated with a given point A (Fig 10) on the unit circle.

SOLUTION. We determine θ to satisfy $(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta = \cos \Theta + i \sin \Theta$, where Θ = amplitude of A. Proceeding as before, we find

$$\theta = \frac{1}{3}\Theta + \frac{2k\pi}{3},$$

whence,

$$k = 0, \qquad \theta_0 = \frac{1}{3}\Theta,$$

$$k = 1, \qquad \theta_1 = \theta_0 + \frac{2\pi}{3},$$

$$k = 2, \qquad \theta_2 = \theta_1 + \frac{2\pi}{3}.$$

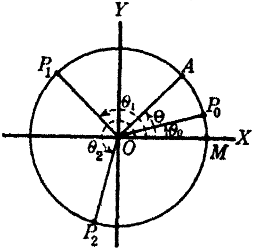

FIG. 10.—The three cube roots of a complex number.

The student will note that after P_0 is found, the point P_1 is located by adding one-third of the circle to the arc MP_0 and the point P_2 is found by adding one-third of the circle to the arc MP_0P_1. *The three cube roots of a given number divide the circle on which they lie into three equal parts.*

The value of Θ is obtained from a table of tangents or it is given; the angle $\theta_0 = \frac{1}{3}\Theta$ is drawn with a protractor.

More generally, to find the n nth roots of a given complex number on the unit circle we must solve for θ the equation

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = \cos\Theta + i\sin\Theta.$$

From this we get

$$n\theta = \Theta + 2k\pi,$$

whence,

$$\theta = \frac{\Theta}{n} + \frac{2k\pi}{n}.$$

As before, one nth root has its amplitude equal to Θ/n. Let it be denoted by θ_0, then $\theta_1 = \theta_0 + \frac{2\pi}{n}$, $\theta_2 = \theta_1 + \frac{2\pi}{n}$, $\cdots$, $\theta_{n-1} = \theta_{n-2} + \frac{2\pi}{n}$. To the angle θ_0 a central angle subtended by an nth part of the circle, *viz.*, $2\pi/n$, has been added n times, determining n points equally spaced on the circle, *i.e.*, determining n values of θ, each of which is the amplitude of a complex number, an nth root of the given complex unit.

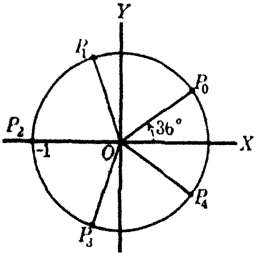

FIG. 11 – The five fifth roots of -1

ILLUSTRATIVE EXAMPLE 3. Find the five fifth roots of -1; *i.e.*, find the solutions of the binomial equation $x^5 + 1 = 0$

SOLUTION The amplitude of -1 is 180°, hence $\theta_0 = 180°/5 = 36°$ One-fifth of the circle is 72°; hence the amplitudes of the five fifth roots of -1 are 36°, 108°, 180°, 252°, 324°. The roots themselves have the form

$$\begin{aligned}
&\cos 36° + i\sin 36°,\\
&\cos 108° + i\sin 108°,\\
&\cos 180° + i\sin 180° = -1,\\
&\cos 252° + i\sin 252°,\\
&\cos 324° + i\sin 324°.
\end{aligned}$$

If the results are desired in Cartesian form, a table of sines and cosines must be used. Only one of the fifth roots of -1 is real.

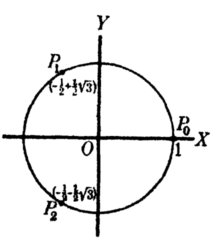
Y
P_1
$(-\frac{1}{2}+\frac{1}{2}\sqrt{3})$
P_0
X
O
1
$(-\frac{1}{2}-\frac{1}{2}\sqrt{3})$
P_2

FIG 12 —The three cube roots of unity.

ILLUSTRATIVE EXAMPLE 4. Find the three cube roots of unity; *i.e.*, find the solutions of the binomial equation $x^3 - 1 = 0$.

SOLUTION. The amplitude of 1 is 0°. The cube roots of 1 are

$$\cos 0° + i\sin 0° = 1,$$

$$\cos 120^\circ + i \sin 120^\circ = -\frac{1}{2} + \frac{i}{2}\sqrt{3},$$
$$\cos 240^\circ + i \sin 240^\circ = -\frac{1}{2} - \frac{i}{2}\sqrt{3}.$$

De Moivre's theorem. General statement. This theorem may be stated for an arbitrary complex number as follows: Let $\rho(\cos\theta + i\sin\theta)$ be any complex number; then $[\rho(\cos\theta + i\sin\theta)]^n = \rho^n(\cos n\theta + i \sin n\theta)$. To apply this to the computation of roots of complex numbers not on the unit circle, we need only note that *if the modulus of the given number is R, the modulus of each of the* nth *roots is equal to the real* nth *root of R.* Hence every nth root will lie on a circle whose radius is $\sqrt[n]{R}$; the amplitudes are determined as before.

ILLUSTRATIVE EXAMPLE 5. Find the four fourth roots of $-16i$.

SOLUTION. We seek values of ρ and θ which will satisfy the equality $\rho^4(\cos 4\theta + i \sin 4\theta) = 16(\cos 270^\circ + i \sin 270^\circ)$; *i.e.*, $\rho^4 = 16$, and $4\theta = 270^\circ + k \cdot 360^\circ$. Therefore $\rho = 2$ and $\theta = 270^\circ/4 + k \cdot 90^\circ$. The four fourth roots of $-16i$ lie on a circle with radius 2; the amplitude of one of the roots is $67\frac{1}{2}^\circ$; the four roots divide the circle into four equal parts. In trigonometric form they are

$2(\cos 67\frac{1}{2}^\circ + i \sin 67\frac{1}{2}^\circ)$, $2(\cos 157\frac{1}{2}^\circ + i \sin 157\frac{1}{2}^\circ)$,
$2(\cos 247\frac{1}{2}^\circ + i \sin 247\frac{1}{2}^\circ)$, $2(\cos 337\frac{1}{2}^\circ + i \sin 337\frac{1}{2}^\circ)$.

EXERCISE V

1. Find the roots of the following **binomial equations**:

(a) $x^4 - 1 = 0$; (b) $x^3 - 8 = 0$;
(c) $x^3 + 8 = 0$; (d) $x^5 - 32 = 0$;
(e) $x^2 + i = 0$; (f) $x^2 = 3 + 2i$.

2. Plot the points associated with the solutions in Problem 1.

8. Properties of the complex number system. Let a, b, c be symbols used to represent given complex numbers. The operations of the complex number system obey the following laws:

1. *Addition is commutative; i.e.,* $a + b = b + a$.
2. *Addition is associative; i.e.,* $(a + b) + c = a + (b + c)$.
3. *Multiplication is commutative; i.e.,* $ab = ba$.
4. *Multiplication is associative; i.e.,* $(ab)c = a(bc)$.
5. *Multiplication is distributive with respect to addition; i.e.,* $c(a + b) = ca + cb$.

Underlying all that is contained in this chapter is the fundamental notion that the result of performing the operation of addition or the operation of multiplication is unique; *i.e.*, there is only one number which is the sum of a finite number of the elements of the system and there is only one number which is the product of a finite number of the elements of the system. Furthermore, with the single exception that zero may not be used as a divisor, the operations of the system may be applied as many times as is desired to any elements of the system whatsoever and the result is an element of the system.

SUMMARY

- The complex number system.
  - Elements.
    - Real numbers.
      - Rational numbers.
        - Positive and negative integers.
        - Zero.
        - Positive and negative rational fractions.
      - Irrational numbers.
    - Complex numbers.
  - Operations.
    - Properties.
- The complex number.
  - Cartesian form.
    - Components.
  - Trigonometric form.
    - Modulus.
    - Amplitude.
    - Radius vector.
  - Conjugate complex numbers.
- Constructions.
  - Sum.
  - Difference.
  - Product.
  - Quotient.
  - Roots.
    - De Moivre's theorem.

CHAPTER II

SOLUTION OF EQUATIONS

In studying the graphs and the properties of rational integral functions of x and of rational functions of x, it was found desirable to be able to determine the zeros of the functions and the zeros of their derivatives. In this chapter we set ourselves two tasks, *viz., to find the rational solutions of a rational integral equation in one variable, and to obtain approximations to all irrational roots.* It is sometimes not difficult and is sometimes quite difficult to determine the imaginary roots, though there is a simple process for discovering how many of the roots of a given equation are imaginary.

9. Solution by formula. Let a rational integral function of x of degree n be denoted by $f(x)$,

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n, \tag{1}$$

where n is a positive integer, and where the coefficients a_0, a_1, a_2, . . . , a_n are *real* numbers with a_0 not zero.

The term a_0x^n is the term of highest degree. It is called the **leading term** and its coefficient a_0 is called the **leading coefficient.** We seek the roots of the equation

$$f(x) = 0. \tag{2}$$

If $n = 1$ the polynomial can have at most two terms. It is called a linear function of x and may be written $a_0x + a_1$.

If $n = 2$, the polynomial becomes the quadratic function $a_0x^2 + a_1x + a_2$, and so on for $n > 2$.

A linear equation $a_0x + a_1 = 0$ has one root (solution) given by the formula

$$x = -\frac{a_1}{a_0}. \tag{3}$$

A quadratic equation $a_0x^2 + a_1x + a_2 = 0$ has two roots given by the quadratic formula* as follows:

$$x_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_0a_2}}{2a_0}, \quad x_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_0a_2}}{2a_0}. \tag{4}$$

---

* Logsdon, "Elementary Mathematical Analysis," Vol. I, p. 172.

Equations (3) and (4) are formulas which contain the coefficients of the linear and quadratic functions, respectively. By their use the solution of any linear equation and the two solutions of any quadratic equation may be written down by merely substituting for the a's and performing only fundamental operations on the numbers of the complex number system, *viz.*, addition, subtraction, multiplication, division, raising to a power, extracting a root. (Chap. I.)

Formulas have also been found by the use of which the three solutions of the general cubic equation $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$ and the four solutions of $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$, the general quartic equation, may be written out by substituting the coefficients and performing the aforementioned operations of arithmetic. These formulas are not given here but may be found in many textbooks on college algebra.*

Quite naturally, after the formulas for the cubic and quartic equations were discovered, mathematicians studied the quintic and equations of higher degree in the hope that they might find other such formulas containing the coefficients which could be used for determining their roots. After many years it was proved in 1824 by the brilliant Norwegian mathematician Abel that *no such formula can be found;* in other words, Abel showed that it is impossible to solve the general n-ic ($n > 4$) by a formula involving only the fundamental operations of arithmetic. There are, however, *methods* by means of which all of the *real* roots of any given polynomial equation may be found (approximately). One of these, Horner's method, is explained in Sec. 15. It was invented about 1819 by an Englishman, W. G. Horner.

10. Relations of roots and coefficients. Though the roots of a general equation of degree $n > 4$ are not expressible in radicals, there are certain simple and interesting relations between the roots and the coefficients of an equation of degree $n \geqq 1$.

Consider, for example, the quadratic equation $a_0x^2 + a_1x + a_2 = 0$ whose roots are indicated notationally by x_1 and x_2. By the factor theorem† the left member of the equation may be written $a_0(x - x_1)(x - x_2)$ which when expanded becomes $a_0x^2 - a_0(x_1 + x_2)x + a_0x_1x_2$. Since the last expression is identical with $a_0x^2 + a_1x + a_2$, the following equalities hold:

---

* See, for example, Hart, "College Algebra," pp. 237 *ff.*, 1926.

† Logsdon, "Elementary Mathematical Analysis," Vol. I, p. 115.

$$x_1 + x_2 = \frac{-a_1}{a_0},\qquad x_1x_2 = \frac{a_2}{a_0}. \tag{5}$$

Thus the sum of the two roots and their product may be written down without first ascertaining the values of the roots themselves.

ILLUSTRATIVE EXAMPLE 1 The sum of the two roots of the equation $6x^2 + 7x - 3 = 0$ is $-7/6$; their product is $-1/2$

ILLUSTRATIVE EXAMPLE 2 The sum and product of the two roots of a quadratic equation are $-14/5$ and $-3/5$, respectively. Write down the equation.

SOLUTION The equation may be written $x^2 + 14/5x - 3/5 = 0$ or $5x^2 + 14x - 3 = 0$

Similarly if the roots of a cubic equation are denoted by x_1, x_2, and x_3, the equation may be written $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$ and also $a_0(x - x_1)(x - x_2)(x - x_3) = 0$. The latter equation when the terms are arranged in descending powers of x is

$$a_0x^3 - a_0(x_1 + x_2 + x_3)x^2 + a_0(x_1x_2 + x_1x_3 + x_2x_3)x - a_0x_1x_2x_3 = 0.$$

A comparison of the original equation with the last equation gives the following equalities:

$$\begin{aligned} x_1 + x_2 + x_3 &= \frac{-a_1}{a_0}, \\ x_1x_2 + x_1x_3 + x_2x_3 &= \frac{a_2}{a_0}, \\ x_1x_2x_3 &= \frac{-a_3}{a_0}. \end{aligned} \tag{6}$$

Let the Greek letter Σ be used to indicate that there is a sum to be taken. The first two of equations (6) may then be written more compactly as follows:

$$\sum_{i=1}^{3} x_i = -\frac{a_1}{a_0},\qquad \sum_{\substack{i,j=1\\ i<j}}^{3} x_ix_j = \frac{a_2}{a_0}.$$

In the first equation the legend $i = 1$ below Σ and 3 above Σ indicates that the i in x_i is to take the values 1, 2, and 3 in turn, after which the terms (here x_1, x_2, and x_3) are to be added. In the second equation i and j may take values 1, 2, and 3 in as many ways as is possible while satisfying the inequality $i < j$.

After all terms $x_i x_j$ which meet these two requirements have been written out they are to be summed. This notation, called the **sigma notation,** is extensively used in mathematics. For example, the sum of all of the even numbers beginning with 12 and ending with 64 is indicated by $\sum\limits_{k=6}^{32} 2k$.

By the fundamental theorem of algebra (see Sec. 11), the general equation of degree n has exactly n roots (not necessarily distinct). Let them be denoted by $x_1, x_2, \ldots, x_n$ corresponding, respectively, to the factors $(x - x_1), (x - x_2), \ldots, (x - x_n)$ of $f(x)$. A comparison of the two ways of writing $f(x)$ gives the following relations:

$$
\begin{aligned}
&\sum_{i=1}^{n} x_i = -\frac{a_1}{a_0}, &\qquad& \sum_{\substack{i,j,k=1\\ i<j<k}}^{n} x_i x_j x_k = -\frac{a_3}{a_0},\\
&\sum_{\substack{i,j=1\\ i<j}}^{n} x_i x_j = \frac{a_2}{a_0}, && \cdots\cdots\cdots\cdots\cdots\cdots\\
& && x_1 x_2 \cdots x_n = (-1)^n \frac{a_n}{a_0}.
\end{aligned}
\tag{7}
$$

The sum of the roots of $f(x) = 0$, as well as the sums of all of the products taken two at a time, three at a time, . . . , n at a time are functions of the coefficients of the powers of x in the given equation.

With the above information at hand it is easy to write down an equation whose roots are given in advance.

ILLUSTRATIVE EXAMPLE 1. Write a quartic equation whose roots are $-2, -\frac{1}{2}, 1, 3$

SOLUTION From equations (7) we get

$$\sum_{i=1}^{4} x_i = -2 - \tfrac{1}{2} + 1 + 3 = \tfrac{3}{2},$$

$$\sum_{\substack{i,j=1\\ i<j}}^{4} x_i x_j = (-2)(-\tfrac{1}{2}) + (-2)\ 1 + (-2)\ 3 + (-\tfrac{1}{2})\cdot 1 + (-\tfrac{1}{2})\cdot 3 + 1\cdot 3 = -6,$$

$$\sum_{\substack{i,j,k=1\\ i<j<k}}^{4} x_i x_j x_k = (-2)(-\tfrac{1}{2})\cdot 1 + (-2)(-\tfrac{1}{2})\cdot 3 + (-2)\cdot 1\cdot 3 + (-\tfrac{1}{2})\cdot 1\cdot 3 = -\tfrac{7}{2},$$

$$x_1 x_2 x_3 x_4 = 3.$$

Hence the coefficients are proportional to 1, $-\frac{3}{2}$, -6, $\frac{7}{2}$, and 3. The required equation may be written $2x^4 - 3x^3 - 12x^2 + 7x + 6 = 0$. The student should check this result by synthetic division and should note that any other equation with -2, $-\frac{1}{2}$, 1, and 3 as roots will be a multiple of this equation; *i.e.*, the equation $2kx^4 - 3kx^3 - 12kx^2 + 7kx + 6k = 0$ has the given roots for any value of k.

ILLUSTRATIVE EXAMPLE 2. Write a cubic equation whose roots are

$$1,\ 2,\ -3.$$

SOLUTION. The sum of the roots is zero; the sum in pairs is -7; the product is -6. Hence the coefficients are proportional to $1, 0, -7, 6$, and we may write the required cubic in the form $x^3 - 7x + 6 = 0$.

Note. *If an equation has a rational fraction as a root, the leading coefficient cannot be unity when the remaining coefficients are integers.*

EXERCISE VI

1. Find an equation whose roots are $1 + i\sqrt{2}$ and $1 - i\sqrt{2}$.

2. Is there a rational integral equation whose roots are $1 + i\sqrt{2}$ and $2 - i\sqrt{2}$?

3. Write equations whose roots are the following:

(*a*) $4, 4, -3$; (*b*) $\frac{1}{2}, \frac{4}{3}, -2, 5$;

(*c*) $2, 0, 5$; (*d*) $0, 0, \frac{2}{3}, -\frac{1}{3}$;

(*e*) $3, 0, -5$; (*f*) $3 - 2i, 3 + 2i, \frac{1+i}{2}, \frac{1-i}{2}$;

(*g*) $1, -1, \frac{1}{5}$; (*h*) $1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$.

4. Derive equations (5) from equations (4).

5. The sum of the roots of a cubic equation is -2, the product is 12, and the difference between the smallest root and the largest root is 7. Find the roots and write the equation.

6. Three roots of a quartic equation are equal; the sum of the four roots is 8, and their product is -27. Find the roots and write the equation.

7. Write the equation whose roots are $2 + i$ and $2 - i$.

8. Write the equation whose roots are i times the roots of the equation in Problem 7.

11. The fundamental theorem of algebra. Equation (3) shows that a linear equation in one variable has one and only one root. Equation (4) shows that a quadratic equation has exactly two roots. These roots are equal when the discriminant of the equation is zero.

More generally, in an equation $f(x) = 0$ of degree n, if $(x - x_1)$ occurs only once as a factor of $f(x)$, the number x_1 is called a **simple root** of the equation; if $f(x)$ contains $(x - x_1)$ as a factor more than once, x_1 is called a **multiple root** of the equation. A multiple root is a double root, a triple root, a quadruple root, etc., when $f(x)$ contains the corresponding linear factor exactly two, three, four, etc., times, respectively. For example, 2 is a triple root of $x^4 - 6x^3 + 12x^2 - 8x = 0$, while $x^4 - 2x^2 + 1 = 0$ has $+1$ as a double root and -1 as a double root.

Geometrically, if $f(x) = 0$ has x_1 as a multiple real root, the tangent to the graph of $y = f(x)$ is horizontal at $x = x_1$; i.e., the derivative $f'(x)$ vanishes at x_1. There are two possibilities as shown by the figures. The proof follows.

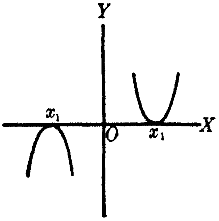

FIG. 13

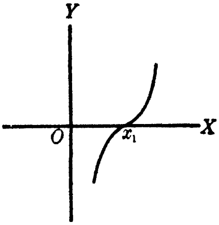

FIG. 14

Let x_1 be a root of $f(x) = 0$ with multiplicity $k > 1$, and let $f(x)$ be written

$$f(x) = (x - x_1)^k F(x),$$

where $F(x)$ does not vanish for $x = x_1$. The derivative of $f(x)$ may be computed by the rule for the derivative of a product. It is

$$\begin{aligned} f'(x) &= (x - x_1)^k F'(x) + k(x - x_1)^{k-1} F(x) \\ &= (x - x_1)^{k-1}[(x - x_1)F'(x) + kF(x)]. \end{aligned}$$

This equation shows that $x = x_1$ is a root of $f'(x)$ with multiplicity exactly $k - 1$, hence at $x = x_1$ the function $y = f(x)$ has a maximum point or a minimum point if k is an even number, and a point of inflexion with a horizontal tangent if k is an odd number.

The **fundamental theorem of algebra** asserts that *a rational integral equation of degree n has exactly n roots when each k-fold root is counted k times.* The proof is made in two steps:

a. Every rational integral equation in a single variable has at least one root.

b. If the degree of a rational integral equation in a single variable is n, the equation has exactly n roots.

In this book no attempt will be made to prove the first statement because the proof involves mathematical knowledge which cannot be included in an elementary text.* The second statement is proved on the assumption that the first is true.

Proof of b. Let the root of $f(x) = 0$ which is guaranteed by statement a be denoted by x_1, and write

$$f(x) = a_0(x - x_1)F_1(x)$$

Now the degree of the polynomial $F_1(x)$ is $n - 1$ and its leading coefficient is unity. If n is greater than 1, by statement a the equation $F_1(x) = 0$ has a root which will be denoted by x_2 Then $F_1(x) = (x - x_2)F_2(x)$ where $F_2(x)$ is a polynomial of degree $n - 2$ in x with unity for its leading coefficient Substituting this value of $F_1(x)$ in the former equation, we get

$$f(x) = a_0(x - x_1)(x - x_2)F_2(x)$$

If n is greater than 2, $F_2(x) = 0$ has a root x_3 by statement a and $(x - x_3)$ is a factor of $F_2(x)$ Consequently

$$f(x) = a_0(x - x_1)(x - x_2)(x - x_3)F_3(x),$$

where the degree of $F_3(x)$ is $n - 3$ This process can be continued for exactly n steps and no more because after n steps the degree of $F_n(x)$ is $n - n = 0$; *i e*, $F_n(x) =$ a constant This constant is unity since the leading coefficient of each of $F_1(x)$, $F_2(x)$, is unity Thus, *because we assumed the existence of at least one root* of $f(x) = 0$ we were able to write $f(x)$ as the product of n linear factors, *viz* ,

$$f(x) = a_0(x - x_1)(x - x_2) \qquad (x - x_n), \tag{8}$$

so that $f(x) = 0$ has n (not necessarily distinct) roots, one root corresponding to each linear factor

No value of x, say, $x = X$, where X is different from $x_1, x_2, \ldots, x_n$ can be a root of $f(x) = 0$, for if so we should have

$$f(X) = a_0(X - x_1)(X - x_2) \qquad (X - x_n) = 0$$

which is impossible since no factor, $X - x_i$, is zero Neither could one of the numbers $x_1, x_2, \ldots, x_n$ be a root of greater multiplicity than is shown in (8), for if so there would be at least one more factor $(x - x_i)$ to be annexed to the right member of (8) This is impossible since the product of more than n linear factors $(x - x_i)$ is a rational integral function of degree more than n and hence could not be equal to $f(x)$ This completes the proof.

Nothing has been said about the numbers $x_1, x_2, \ldots, x_n$ which are the n roots of $f(x) = 0$ except that they are not necessarily all distinct. Some or all of them may be integers, as 2, 5, 0, etc., they may be rational fractions, as $\frac{1}{2}$, $-\frac{5}{3}$, etc., they may be irrational algebraic numbers, as $\sqrt{3}$, $1 + \sqrt[5]{5}$, 1. 478 . . . , etc., or they may be complex numbers, as $1 + 2i$ and $1 - 2i$ which are the (conjugate) roots of $x^2 - 2x + 5 = 0$. The problem of obtaining the complex roots of $f(x) = 0$ when there is no formula for the roots (*i.e.*, for $n > 4$) is more difficult than the problem of finding the real roots. There is, however, a theorem concerning complex (imaginary) roots which is frequently very useful. It will be proved now.

---

* See Dickson, "A First Course in the Theory of Equations," p 155. The first proof of the fundamental theorem of algebra was made by Gauss, a German mathematician, in 1799

Theorem. *There is always an even number of imaginary roots of the rational integral equation* $f(x) = 0$.

Proof. If $f(x) = 0$ has no imaginary root, the theorem is true since zero is an even number. If there is one imaginary root, $r + si$, we shall prove that $r - si$ is also a root. Let $r + si$ be a root, then we have

$$f(r + si) = a_0(r + si)^n + a_1(r + si)^{n-1} + \cdots + a_n = 0.$$

Let every binomial $(r + si)^k$ be expanded (by multiplication) and let i^2, i^4, etc., be replaced by -1, 1, etc. There will be terms containing i and terms not containing i. Denote by $R + Si$ the complex number obtained when like terms have been combined. Then

$$f(r + si) = R + Si = 0,$$

from which it follows from the definition of equality on page 6 that $R = 0$ and $S = 0$.

We wish to prove $f(r - si) = 0$, where

$$f(r - si) = a_0(r - si)^n + a_1(r - si)^{n-1} + \cdots + a_n.$$

Now $r - si$ is obtained from $r + si$ by replacing i by $-i$; consequently we may obtain $f(r - si)$ by expanding the right member of the last equation or by replacing i by $-i$ in the expansion of $f(r + si)$. Such a replacement will not affect terms in $i^2, i^4, \cdots$, since $(-i)^2 = i^2$, $(-i)^4 = i^4$, etc., nor will it affect terms which contain no power of i. When all of these terms, *viz.*, constants, i^2 terms, i^4 terms, etc., are added the sum will be R as before. When i is replaced by $-i$ in a term containing an odd power of i, the sign of the term is changed, since $(-i)^3 = -i^3$, $(-i)^5 = -i^5$, etc. The sum of these terms will then be $-Si$. Thus $f(r - si) = R - Si$, but since it was shown above that $R = 0$ and $S = 0$, we have $f(r - si) = 0$. *q.e.d.*

Cor. 1. *The imaginary roots of* $f(x) = 0$ *may be grouped in conjugate imaginary pairs.*

Cor. 2. *Corresponding to a pair of conjugate complex roots of* $f(x) = 0$ *will be a quadratic factor of* $f(x)$ *with real coefficients*, for, the numbers $r + si$ and $r - si$ are roots of the quadratic equation $x^2 - 2rx + r^2 + s^2 = 0$. Such a quadratic expression is called **irreducible** since it can not be factored into linear factors with real coefficients.

12. Rational roots. There are direct and comparatively simple processes for finding all of the *rational* roots of $f(x) = 0$. After a rational root, say, $x = x_1$, has been found, computation of other roots is simplified if the linear factor $(x - x_1)$ is factored out. This is done by synthetic division. We may write

$$f(x) = (x - x_1)F_1(x). \tag{9}$$

The simplification is due to the fact that all of the roots of $F_1(x) = 0$ are roots of $f(x) = 0$, since any value of x for which $F_1(x)$ vanishes will by equation (9) make $f(x)$ vanish also. Moreover, since $F_1(x) = 0$ has $n - 1$ roots, these with x_1 constitute the n roots of $f(x)$. Obviously it will be simpler to continue the work with an equation of degree $n - 1$ than with an equation of degree n. The equation $F_1(x) = 0$ is called the **depressed equation.**

As fast as we discover rational roots we shall depress the last equation used. When we have found all of the rational roots and have removed the corresponding linear factors, the depressed equation is now an equation which has *complex* roots or *irrational* roots or both. The next step is to find approximately the irrational roots, if any, using the last depressed equation. Thus the tedious process of Horner's method is applied with the minimum amount of labor.

The three theorems which furnish the means of discovering the rational roots of $f(x) = 0$ where the coefficients of $f(x)$ are *integers* are as follows:

Theorem I. *An integer r which is a root of $f(x) = 0$ is a factor of the constant term a_n.*

Theorem II. *If the leading coefficient of $f(x)$ is unity when the remaining coefficients are integers, any rational root of $f(x) = 0$ must necessarily be an integer.*

Theorem III. *If when the leading coefficient of $f(x)$ is unity the remaining coefficients are not all integers, a new equation $F(x) = 0$ can be found which satisfies the conditions of Theorem II and whose roots are proportional to the roots of $f(x) = 0$.*

In what follows, it is assumed that the coefficients $a_0, a_1, \ldots, a_n$ are not fractions. If so, the equation can be multiplied by the least common multiple of all the denominators without disturbing the roots.

Proof of Theorem I. Let the integer r which is assumed to be a root of $f(x) = 0$ be substituted for x in the equation. The equation is satisfied for $x = r$; *i.e.*,

$$a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r = -a_n.$$

The left member of this equality is divisible by r; hence the right member is also divisible by r. *q.e.d.*

ILLUSTRATIVE EXAMPLE. Find the integral roots of

$$2x^4 + 5x^3 - 4x^2 - 4x - 3 = 0.$$

SOLUTION. By Theorem I if there are integral roots they are factors of 3. Hence it is only necessary to try $+1$, -1, $+3$, and -3 Computation (by synthetic division) gives

$$\begin{array}{ll}
f(1) = -4, & \\
f(-1) = -6, & \\
f(3) = 246, & \\
f(-3) = 0 &
\end{array}
\qquad
\begin{array}{r|l}
2 + 5 - 4 - 4 - 3 & \underline{-3} \\
\underline{\;\; - 6 + 3 + 3 + 3} & \\
2 - 1 - 1 - 1 \quad\;\; &
\end{array}$$

Hence there is only one *integral* root; it is $x = -3$. The synthetic division gives $2x^4 + 5x^3 - 4x^2 - 4x - 3 = (x + 3)(2x^3 - x^2 - x - 1)$ The depressed equation is $2x^3 - x^2 - x - 1 = 0$; it has no integral roots since the only possible integral roots are $+1$ and -1 and these were shown not to be roots of $f(x) = 0$; hence they cannot be roots of any equation found by setting a factor of $f(x)$ equal to zero

EXERCISE VII

Find the integral roots, if any, of the following equations:

(a) $x^4 - x^3 - 2x^2 - 9x - 9 = 0$
(b) $x^4 + 7x^3 + 17x^2 + 26x + 24 = 0$
(c) $x^3 - x^2 - 14x + 8 = 0$
(d) $3x^4 - 2x^3 - 49x^2 - 76x - 20 = 0$
(e) $x^3 + 5x^2 - 16x - 80 = 0$
(f) $x^4 + 4x^3 + 7x^2 + 6x = 0$
(g) $x^3 + 3x^2 - 5x - 4 = 0$
(h) $x^3 + 6x^2 + x + 6 = 0$

Proof of Theorem II. We assume that the equation $f(x) = 0$ has a rational fraction in its lowest terms, say r/s, as a root and prove that s must be unity; this proves that there cannot be a rational solution which is not an integer if the leading coefficient of $f(x)$ is unity when the remaining coefficients are integers.

Suppose s is not unity and substitute r/s for x, afterwards multiplying the two members by s^{n-1}. This gives

$$\frac{r^n}{s} + a_1r^{n-1} + a_2r^{n-2}s + \cdots + a_{n-1}rs^{n-2} + a_nrs^{n-1} = 0.$$

Now if s is not unity the first term is not an integer, hence it is impossible to get zero by adding r^n/s to the integers a_1r^{n-1}, $a_2r^{n-2}s$, . . . But the sum must be zero because r/s is a root. Hence r^n/s is an integer, and $s = 1$. *q.e.d.*

These two theorems enable us to find *all of the rational roots* of $f(x) = 0$ if the leading coefficient is unity and each coefficient is a whole number, but if the leading coefficient is not unity when the remaining coefficients are whole numbers, we have from the theorems no method for finding rational, non-integral roots. Theorem III furnishes exactly what is needed to enable us *to determine all of the rational roots of any given rational integral equation with rational coefficients.* The procedure will be illustrated by an example.

ILLUSTRATIVE EXAMPLE. Find the rational roots of

$$f(x) = 16x^3 + 44x^2 + 4x - 15 = 0.$$

SOLUTION. By trial it is found that no factor of 15 satisfies the equation; hence there is no integral root.*

Let the given equation be divided by 16 to make the leading coefficient unity. There is a possibility that there may be fractional roots. We get

$$x^3 + {}^{11}\!/_{4}x^2 + {}^{3}\!/_{8}x - {}^{15}\!/_{16} = 0. \tag{10}$$

In (10) we put $x = x'/k$, after which we multiply each term of the new equation by k^3. This gives

$$x'^3 + {}^{11}\!/_{4}kx'^2 + {}^{3}\!/_{8}k^2x' - {}^{15}\!/_{16}k^3 = 0. \tag{11}$$

The symbol k is the, as yet unknown, *factor of proportionality* mentioned in the theorem. We may choose for k any value which will make the coefficients of (11) all integers. The roots of (11) which are rational will

---

* It is only necessary to try +1, −1, and −3. Substitution of +1 gives all positive signs in the quotient and a positive remainder. Obviously substitution of a positive number greater than +1 would never produce zero for the remainder term, since each term after the first not only would be positive but would be larger than the corresponding term appearing when +1 is substituted. The number 1 is in this case said to be an **upper limit of the roots.**

$$\begin{array}{rrrr|l} 16 & +44 & +4 & -15 & \underline{1} \\ & \underline{+16} & \underline{+60} & \underline{+64} & \\ 16 & +60 & +64 & +49 & \end{array}$$

Similarly, when −3 is substituted the signs of the terms are alternately plus and minus. Consider the effect of substituting a negative number numerically larger than 3; the first product will be negative and numerically larger than 48, the sum will be negative and numerically larger than 4; the next product will be positive and greater than 12, the sum will be positive and greater than 16; etc. Thus, if when substituting a negative number, $-r$, the terms are alternately plus and minus, it is impossible to get a zero remainder by substituting a negative number numerically greater than r. The number $-r$ is a **lower limit of the roots.**

$$\begin{array}{rrrr|l} 16 & +44 & +4 & -15 & \underline{-3} \\ & \underline{-48} & \underline{+12} & \underline{-48} & \\ 16 & -4 & +16 & -63 & \end{array}$$

By being on the alert for these two situations the student will save himself the labor of substituting numbers which are too large or too small to be solutions of the given equation.

then be integers and factors of the constant term. These roots of (11) can be found by trial as shown above. If they are denoted by x_1', x_2', x_3', the roots of (10) are x_1'/k, x_2'/k, x_3'/k since x' of (11) and x of (10) are related by $x'/k = x$.

The value of k in (11) may be 4 or any multiple of 4, as may be seen by inspection. The subsequent computation is simplest if we choose the smallest value of k which will serve, *viz.* 4; this gives from (11)

$$x'^3 + 11x'^2 + 4x' - 60 = 0. \tag{12}$$

By synthetic division 2, -3, and -10 are found to be roots of (12); hence the three roots of the original equation are $\frac{1}{2}$, $-\frac{3}{4}$, and $-\frac{5}{2}$.

This process of finding a new equation whose roots are k times the roots of a given equation is called **multiplying the roots of the equation by *k*.** From what precedes we can formulate the rule as follows:

To multiply the roots of a given equation by k, arrange the terms of the equation in order, a_0x^n, a_1x^{n-1}, . . . , a_n; *multiply* a_1 *by* k, a_2 *by* k^2, a_3 *by* k^3, . . . , a_n *by* k^n.

Zero coefficients must not be overlooked.

EXERCISE VIII

1. Multiply the roots of each of the following equations by the number at its right:

(*a*) $3x^4 - 4x^3 - 3x^2 + 5x + 1 = 0$ 3.
(*b*) $2x^5 + 7x^3 + 3x - 7 = 0$ 2.
(*c*) $5x^3 - 3x^2 - 2x + 8 = 0$ 5.
(*d*) $4x^4 + x^2 - 5x + 6 = 0$ 2.
(*e*) $3x^4 - 3x^3 + 6x^2 + 5x + 2 = 0$ -1.
(*f*) $x^3 + 1 = 0$ -1.
(*g*) $4x^4 + 7x^2 + 1 = 0$ 2.
(*h*) $4x^4 + 7x^2 + 1 = 0$ 4.
(*i*) $4x^4 + 7x^2 + 1 = 0$ -1.
(*j*) $8x^6 + 1 = 0$ 2.

2. The roots of the equation $8x^3 - 28x^2 + 22x - 5 = 0$ are $\frac{1}{2}$, $\frac{1}{2}$, and $\frac{5}{2}$. Multiply the roots by 2 and check the result by showing that the transformed equation has 1, 1, and 5 for its roots.

3. Transform the equation in Problem 2 into another equation whose roots have the values $-\frac{1}{2}$, $-\frac{1}{2}$, and $-\frac{5}{2}$.

4. Transform the equation in Problem 3 into another equation whose roots have the values -1, -1, and -5.

5. Write the equation whose roots are -5, -3, and -2. By the method in the last paragraph, transform it into another equation whose roots are $-\frac{1}{2}$, $-\frac{3}{10}$, and $-\frac{1}{5}$, *i.e.*, divide the roots of the given equation by 10. The new equation has three roots between 0 and -1.

6. By what number may the roots of the following equations be multiplied in order that the resulting equation may have its leading coefficient unity and the remaining coefficients integers? Write the transformed equations.

(*a*) $3x^3 - 2x + 9 = 0$; (*b*) $4x^4 + 3x^3 - 12 = 0$;
(*c*) $4x^4 + 8x^3 - 6x + 1 = 0$; (*d*) $4x^4 + 3x^2 + 12 = 0$;
(*e*) $6x^3 - 5x^2 + 12 = 0$; (*f*) $25x^4 - 7x + 50 = 0$.

7. Find all of the rational roots of the following equations:

(*a*) $x^6 - 5x^2 - 4x + 20 = 0$;
(*b*) $2x^3 - 11x^2 + 17x - 6 = 0$;
(*c*) $6x^3 - 7x^2 - 18x - 5 = 0$;
(*d*) $12x^3 - 20x^2 - x + 6 = 0$;
(*e*) $12x^3 - 32x^2 + 44x + 15 = 0$;
(*f*) $9x^4 + 27x^3 + 5x^2 - 27x - 14 = 0$;
(*g*) $x^4 - 9x^3 + 11x^2 + 27x + 90 = 0$;
(*h*) $x^4 + 5x^2 + 4 = 0$;
(*i*) $x^3 + 6x^2 + 6x + 36 = 0$;
(*j*) $3x^4 + 4x^3 + 3x^2 + 4x = 0$.

13. Some geometric properties. There are certain geometric interpretations of the relations discussed in the preceding paragraphs and in Vol. I of this series which it will be profitable to collect and tabulate before taking up Horner's method for irrational roots. Associated with the equation $f(x) = 0$ is the function $y = f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$; the following are some properties of its graph.

1. It is continuous and single valued for all values of x, becoming infinite only when x gets large without bound. (See Vol. I, page 123.)

2. It may have maximum points, minimum points, or points of inflexion with horizontal tangents. The abscissas of such points are found by solving the equation $f'(x) = 0$.

3. The totality of maximum and minimum points cannot exceed $n - 1$, since the degree of $f'(x)$ is $n - 1$ and consequently the equation $f'(x) = 0$ can have at most $n - 1$ real roots.

4. When the graph of $y = f(x)$ does not have a totality of $n - 1$ maximum and minimum points, the number which it possesses differs from $n - 1$ by an even number. This is due to the fact that if $f'(x) = 0$ has complex roots it has an even number of them.

5. When the roots of $f(x) = 0$ are multiplied by k (*cf.* equation (11), page 30) the effect on the graph of $y = f(x)$ is to stretch it horizontally if k is greater than unity, thus separating the roots more than before.

6. If the function is positive for $x = a$ and negative for $x = b$ (or conversely), because it is a continuous function it is zero for at least one value of x which lies between a and b. Geometrically, if the curve is on opposite sides of the x-axis at $x = a$ and $x = b$, it crosses the x-axis at least once between a and b. This point is called a zero of the function and its abscissa is a root of the equation $f(x) = 0$. For example, the zeros of the function $y = x^3 - x^2 - 6x$ whose graph is the heavy curve in Fig. 15 are at A, O, and B and correspond respectively to the roots $x = -2$, $x = 0$, and $x = 3$ of the equation $x^3 - x^2 - 6x = 0$.

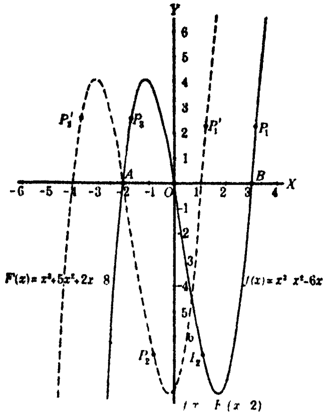

FIG. 15.—Diminishing the roots of an equation.

Let the cubic curve in Fig. 15 be moved, say, two units to the left of its original position. Its new position is indicated in the figure by the dotted curve. It is desired to find the polynomial function, denoted by $y = F(x)$, whose graph is the dotted curve in Fig. 15. Let the unknown coefficients of $F(x)$ be denoted by b_0, b_1, b_2, b_3, as follows:

$$F(x) = b_0x^3 + b_1x^2 + b_2x + b_3.$$

The values of the coefficients in $F(x)$ can be obtained by noticing that for points on $y = f(x)$, as P_1, P_2, P_3, with abscissas x_1, x_2, x_3, respectively, there are on the graph of $y = F(x)$ points with equal ordinates but with abscissas $x_1 - 2$, $x_2 - 2$, $x_3 - 2$, respectively. More generally, for *any point* P with abscissa x on $y = f(x)$ there is on $y = F(x)$ a corresponding point P' with abscissa $x - 2$ and with ordinate equal to the ordinate of P; *i.e.*,

$$F(x - 2) = f(x) \tag{13}$$

But $F(x - 2) = b_0(x - 2)^3 + b_1(x - 2)^2 + b_2(x - 2) + b_3$, and consequently, by (13),

$$f(x) = b_0(x - 2)^3 + b_1(x - 2)^2 + b_2(x - 2) + b_3. \tag{14}$$

The two members of (14) are merely two ways of writing the same polynomial, consequently when any value is substituted for x, the numbers obtained on the two sides of (14) are equal. In particular, we set $x = 2$ and get

$$f(2) = b_3.$$

Thus the value of one of the unknown coefficients of $F(x)$ has been obtained. It is most simply computed by the method of synthetic division, since when $f(x)$ is divided by $x - 2$ the remainder is $f(2)$.

Let us now again divide both members of equation (14) by $x - 2$; not only will the remainders be equal but also the quotients. On the right the quotient is $b_0(x - 2)^2 + b_1(x - 2) + b_2$. A second division by $x - 2$ gives b_2 as the remainder and $b_0(x - 2) + b_1$ as the quotient. The corresponding division on the left must also give b_2 as the remainder and the same quotient.

A third division by $x - 2$ of the last quotient on the right gives b_1 as the remainder and b_0 as the quotient. Thus we have found the following method of obtaining the coefficients of $F(x)$.

1. *Divide $f(x)$ by $x - 2$; the remainder is b_3.*

2. *Divide by $x - 2$ the* **quotient** *obtained by the first division; the remainder is b_2.*

3. *Divide by $x - 2$ the quotient obtained by the second division; the remainder is b_1 and the quotient is b_0.* No further division is possible, after three divisions have been performed.

The computation for the function whose graph is the dotted curve in Fig. 15 is now shown.

$$f(x) = x^3 - x^2 - 6x \qquad \begin{array}{rrrrl} 1 & -1 & -6 & +0 & |\underline{2} \\ & +2 & +2 & -8 & \\ \hline 1 & +1 & -4 & |-8 & \\ & +2 & +6 & | & \\ \hline 1 & +3 & |+2 & & \\ & +2 & | & & \\ \hline 1 & +5 & & & \end{array} \qquad \begin{array}{l} b_3 = -8 \\ b_2 = 2 \\ b_1 = 5 \\ b_0 = 1 \end{array}$$

$$F(x) = x^3 + 5x^2 + 2x - 8.$$

From its origin we see that the graph of the function $y = F(x)$ *has zeros two units to the left of the zeros of* $y = f(x)$. In other words the equation $x^3 + 5x^2 + 2x - 8 = 0$ has roots which are smaller by 2 than the roots of $x^3 - x^2 - 6x = 0$. The roots of the latter were found to be -2, 0, and 3; consequently the roots of the new equation should be $x = -4, -2$, and 1. This result should be checked by the student.

The process of continued division by $(x - 2)$ gave an equation whose roots are the roots of the given equation *diminished by* 2.

In the same way continued division by $(x + 3)$ gives an equation whose roots are the roots of the given equation diminished by (-3), *i.e.*, increased by 3 This corresponds geometrically to moving the graph three units to the right The process is called **diminishing the roots of an equation.**

ILLUSTRATIVE EXAMPLE 1 Diminish by 3 the roots of

$$2x^3 - x^2 - 32x + 16 = 0$$

SOLUTION We divide by $x - 3$ by substituting $+3$ synthetically

$$\begin{array}{rrrr|l}
2 & -\ 1 & -\ 32 & +\ 16 & \underline{3} \\
 & +\ 6 & +\ 15 & -\ 51 & \\
\hline
2 & +\ 5 & -\ 17 & -\ 35 & \\
 & +\ 6 & +\ 33 & & \\
\hline
2 & +\ 11 & +\ 16 & & \\
 & +\ 6 & & & \\
\hline
2 & +\ 17 & & &
\end{array}$$

$$F(x) = 2x^3 + 17x^2 + 16x - 35 = 0$$

Check The roots of the first equation are -4, $\frac{1}{2}$, and 4, the roots of the new equation are -7, $-\frac{5}{2}$, and 1

ILLUSTRATIVE EXAMPLE 2 Diminish by -4 the roots of

$$2x^3 - x^2 - 32x + 16 = 0$$

SOLUTION

$$\begin{array}{rrrr|l}
2 & -\ 1 & -\ 32 & +\ 16 & \underline{-4} \\
 & -\ 8 & +\ 36 & -\ 16 & \\
\hline
2 & -\ 9 & +\ 4 & +\ 0 & \\
 & -\ 8 & +\ 68 & & \\
\hline
2 & -\ 17 & +\ 72 & & \\
 & -\ 8 & & & \\
\hline
2 & -\ 25 & & &
\end{array}$$

The desired equation $2x^3 - 25x^2 + 72x = 0$ has roots which are greater by 4 than the corresponding roots of the given equation

EXERCISE IX

From the following equations obtain new equations whose roots are the roots of the given equation diminished in each case by the number in the parentheses

(*a*) $2x^4 - 3x^3 - 5x^2 - 8x + 10 = 0$ (3)
(*b*) $x^3 - 5x^2 - 7 = 0$ (4)
(*c*) $2x^3 + 7x^2 + 11x - 1 = 0$ (−3)
(*d*) $4x^4 + 7x^2 - 11 = 0$ (−1)
(*e*) $2x^3 - 5x + 3 = 0$ (1)
(*f*) $x^3 + 3x^2 + 4x - 9 = 0$ (0 1)
(*g*) $4x^4 + 2x^2 + 5 = 0$ (0 05)

14. Locating irrational roots. We can assume in this section that if a given equation has rational roots they have all been determined and the equation has been depressed (see page 27) by one unit for each rational root, so that $f(x) = 0$ can have only irrational or imaginary roots. If there are irrational roots, the first step is to try to find between what two consecutive integers they lie. For example, let us study the equation

$$x^3 - 5x^2 + 4x + 5 = 0 \qquad (15)$$

to see if it has irrational roots. The appended table of values of the function $y = x^3 - 5x^2 + 4x + 5$ is computed.

x	y
−1	−5
→	
0	5
1	5
2	1
→	
3	−1
→	
4	5
5	25

An arrow in the table directs the attention to the fact that the function has changed sign; hence, by property 6 on page 32, the three roots of the cubic equation (15) are located as follows:

(1) one positive root between 2 and 3,
(2) one positive root between 3 and 4,
(3) one negative root between −1 and 0.

Since a cubic has exactly three roots, this cubic has no complex roots.

EXERCISE X

Locate the irrational roots of the following equations:

(*a*) $x^4 - 3x^3 + 5 = 0$;
(*b*) $x^3 - 3x + 7 = 0$;
(*c*) $x^3 + 5x^2 + 2 = 0$;
(*d*) $2x^3 + 3x^2 - 5x + 6 = 0$;
(*e*) $x^3 - 2x^2 - 2x - 4 = 0$;
(*f*) $2x^4 - 3x^3 - 5x^2 + 7 = 0$;
(*g*) $3x^4 - 5x^2 - 1 = 0$;
(*h*) $x^5 + 3x^4 + 5x^2 - 8x - 10 = 0$.

ILLUSTRATIVE EXAMPLE 2. Locate the irrational roots, if any, of

$$x^4 + 1 = 0.$$

x	y
−2	17
−1	2
0	1
1	2
2	17

SOLUTION. The table of values for $y = x^4 + 1$ need not be carried far. Since there is no odd power of x in the right member, y has the same value when, say, $-a$ is substituted as when $+a$ is substituted. This means that for any point on the graph which lies to the right of the y-axis, there is a corresponding point on the left of the y-axis, and that these two points are symmetric with respect to the y-axis. (*Cf.* Vol. I, pages 37 and 165.)

The table of values of the function $x^4 + 1$ leads us to suspect that the function has no real zero. The derivative of y with respect to x is $4x^3$. This is zero only when $x = 0$. Hence the *minimum value of the function* (the table shows that there is a minimum value) is its value when $x = 0$; *viz.*, $y = 1$. Hence the function has no real zero and the equation $x^4 + 1 = 0$ has no real root.

Its four roots are imaginary and by Cor. 1 on page 26 they form two conjugate pairs. They can be obtained algebraically as follows:

$$\begin{aligned} x^4 + 1 &= 0, \\ x^4 &= -1, \\ x^2 &= \pm i, \\ x &= \pm \sqrt{i}, \pm \sqrt{-i}. \end{aligned}$$

These roots can be expressed trigonometrically by the method of Sec. 7, Chap I.

Note. It is evident that an equation with no real root is of even degree, and that an equation whose degree is odd must have at least one real root. Geometrically, a cubic curve, quintic curve, etc , crosses the x-axis *at least once.*

ILLUSTRATIVE EXAMPLE 3 Locate the real roots of

$$128x^3 - 216x^2 + 108x - 15 = 0.$$

SOLUTION. From the table of values of the function

$$y = 128x^3 - 216x^2 + 108x - 15$$

we conclude that there is *at least* one real root between 0 and 1. Moreover, the synthetic division shows thar 2 is an upper limit and -1 a lower limit of the roots.

x	y
-1	-467
0	-15
→	
1	$+5$
2	$+361$

We cannot conclude that the remaining two roots are not real, for it is possible for the curve to cross the x-axis twice between two consecutive digits or three times between 0 and 1. We examine the derivative in order to find the maximum and minimum points (if any) of the given cubic. The derivative is $12(32x^2 - 36x + 9)$; its zeros are $x = 3/8$ and $x = 3/4$. We include these values in the table of values; the conclusion is that there are three real roots between 0 and 1.

x	y
0	-15
→	
$3/8$	$+15/8$
→	
$3/4$	$-3/2$
→	
1	3

There is no danger of not finding all of the roots of a given equation if all of its turning points are located and included in the table of values.

EXERCISE XI

Find the rational roots, locate the irrational roots, and when possible determine the number of complex roots of the following equations:

(a) $x^3 - 5x + 5 = 0$; (b) $4x^4 + 2x^3 - 6x^2 + 11x + 7 = 0$;
(c) $x^4 - 6x^2 + 9 = 0$; (d) $6x^3 - 7x^2 - 16x + 12 = 0$;
(e) $5x^3 - 10x^2 + 2 = 0$; (f) $x^4 + 2x^3 + 7x^2 + 6x + 12 = 0$;
(g) $x^4 - 6x^3 + 9 = 0$; (h) $6x^3 - 5x^2 - 17x + 6 = 0$;
(i) $x^5 + 32 = 0$; (j) $6x^4 + 13x^3 + 7x^2 + 26x - 10 = 0$;
(k) $x^4 + x^2 + 1 = 0$; (l) $9x^4 - 36x^3 + 29x^2 + 62x + 16 = 0$.

15. Horner's method for irrational roots. To illustrate the method, let us compute the irrational root of

$$x^3 - 5x^2 + 4x + 5 = 0. \tag{16}$$

On page 35 it was found that a root of this equation lies between 2 and 3. We diminish the roots of (16) by 2 and obtain a new equation,

$$x^3 + x^2 - 4x + 1 = 0 \tag{17}$$

which has a root *between zero and* 1.

$$\begin{array}{rrrr|l}
1 & -5 & +4 & +5 & \underline{2} \\
 & +2 & -6 & -4 & \\
\hline
1 & -3 & -2 & +1 & \\
 & +2 & -2 & & \\
\hline
1 & -1 & -4 & & \\
 & +2 & & & \\
\hline
1 & +1 & & &
\end{array}$$

By synthetic division we try 0.1, 0.2, 0.3, finding that the remainder is positive for $x = 0$, 0.1, and 0.2 but is negative for $x = 0.3$; *i.e.*, equation (17) has a root between 0.2 and 0.3. The next step is to diminish the roots by 0.2, thus obtaining a new equation which has a root *between zero and* 0.1.

$$\begin{array}{rrrr|l}
1 & +1 & -4 & +1 & \underline{0.3} \\
 & +0.3 & +0.39 & -1.083 & \\
\hline
1 & +1.3 & -3.61 & -0.083 &
\end{array}
\qquad
\begin{array}{rrrr|l}
1 & +1 & -4 & +1 & \underline{0.2} \\
 & +0.2 & +0.24 & -0.752 & \\
\hline
1 & +1.2 & -3.76 & +0.248 & \\
 & +0.2 & +0.28 & & \\
\hline
1 & +1.4 & -3.48 & & \\
 & +0.2 & & & \\
\hline
1 & +1.6 & & &
\end{array}$$

The new equation is

$$x^3 + 1.6x^2 - 3.48x + 0.248 = 0. \tag{18}$$

An *approximation* of the desired root of (18) is obtained by neglecting all of the terms except the linear term and constant; *viz.*,

$$-3.48x + 0.248 = 0,$$
$$x = 0.07 \text{ (approx.)}.$$

Since x is less than one-tenth, x^2 and x^3 will be small in comparison with x. When the terms in x^2 and x^3 are neglected, we get an approximation whose accuracy depends partly on the actual size of the root and partly on the sizes and signs of the coefficients of the powers of x which are neglected. In general, the smaller the root sought, the more reliable will be the approximation obtained by neglecting the higher powers of x.

In the above example, we try 0.07 by synthetic division and find a remainder of $+0.012583$; we then try 0.08 and find a remainder of -0.019648. Hence the actual value of the root is greater than 0.07 and less than 0.08. The roots of (18) are next diminished by 0.07 and a new equation,

$$x^3 + 1.81x^2 - 3.2413x + 0.012583 = 0, \qquad (19)$$

is obtained. This equation has a root which lies between zero and 0.01.

An approximation is obtained by solving

$$-3.2413x + 0.012583 = 0,$$
$$x = 0.00388.$$

It can be shown that when there are k zeros after the decimal point the best approximation is obtained by taking k additional digits in the quotient. Here, then, we can say the root is 0.0039.

If greater accuracy is desired, diminish the roots of (19) by 0.003, after which the approximation will have (at least) three zeros after the decimal place and may consequently be accepted to the sixth place. In fact it is 0.000886.

Going backward from (19) to the original equation, the results may be tabulated as follows:

a. A root of (19) is 0.003886 (or .0039);
b. The corresponding root of (18) is 0.073886;
c. The corresponding root of (17) is 0.273886;
d. The corresponding root of (16) is 2.273886.

By continuing the process a suitable number of times, as many decimal places may be obtained as are desired.

The graphs of all of the functions, the left members of equations (16), (17), (18), and (19) may be traced from a pattern cut from any one of them. All that is necessary is to move the pattern on the x-axis until it is in the proper position with respect to its zeros.

Equation (16) has a negative root between -1 and 0. (See page 35.) It is more awkward to work with negative than with positive roots. Let us multiply the roots of equation (16) by (-1) by the method of Sec. 13. We have

$$f(x) = x^3 - 5x^2 + 4x + 5 = 0, \qquad (16)$$
$$-f(-x) = x^3 + 5x^2 + 4x - 5 = 0, \qquad (20)$$

The negative root of (16) which lies between 0 and -1 corresponds to a positive root of (20) which lies between 0 and $+1$. The numerical value of the root is not changed by multiplying it by (-1).

Successive trials show that the desired root of (20) lies between 0.6 and 0.7. When the roots are diminished by 0.6 the resulting equation is

$$x^3 + 6.8x^2 + 11.08x - 0.584 = 0,$$

which has a root between zero and one-tenth. The approximation is given by

$$\begin{aligned} 11.08x &= 0.584, \\ x &= 0.05. \end{aligned}$$

Trial shows that the root lies between 0.05 and 0.06. We diminish the roots by 0.05, obtaining the new equation

$$x^3 + 6.95x^2 + 11.7675x - 0.012875 = 0,$$

which has a positive root whose value is less than 0.01.

$$\begin{array}{llll|l}
1 & +6\ 8 & +11\ 08\ . & -0\ 584 & \underline{0\ 05} \\
 & 0\ 05 & +\ \ 0\ 3425 & +0\ 571125 & \\
\hline
1 & +6\ 85 & +11\ 4225 & -0\ 012875 & \\
 & 0\ 05 & +\ \ 0\ 3450 & & \\
\hline
1 & +6\ 90 & +11\ 7675 & & \\
 & 0\ 05 & & & \\
\hline
1 & +6\ 95 & & &
\end{array}$$

The approximation to the root is given by solving $11.7675x = 0.012875$; it is $x = .00109$. Consequently, the best approximation to four digits to the root under discussion of $f(-x) = 0$ is 0.6511; the corresponding negative root of equation (16) is $x = -0.6511$.

EXERCISE XII

1. Find all of the real roots of

(a) $x^3 - 2x^2 - 5x + 7 = 0$;
(b) $2x^3 - 7x^2 - 30 = 0$;
(c) $x^4 + 4x^3 - 9x^2 - 120 = 0$;
(d) $x^3 + x^2 - 7x - 3 = 0$;
(e) $x^4 - 11x^3 + 38x^2 - 51x + 27 = 0$;
(f) $x^3 - 9 = 0$;
(g) $32x^3 - 80x^2 - 66x + 189 = 0$;
(h) $x^4 - 8 = 0$.

2. A box 18 in. long, 10 in. wide, and 8 in. deep is to be doubled in volume by increasing each dimension by x in. Find the value of x to three decimal places.

3. A spherical shell has an inner radius of 12 in.; the volume of the solid part is 600 cu. in. Find the thickness of the shell.

4. A sector is to be cut from a circular piece of cloth and fashioned into a conical tent. The radius of the circle is 9 ft. and the cubic contents of the tent must be 10π cu. ft. Find the radius of the base of the tent and the angle α of the sector.

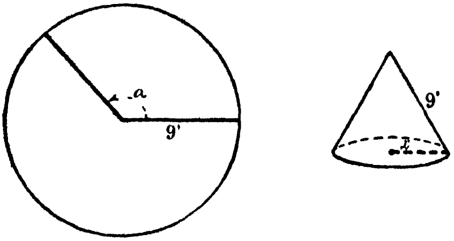

FIG. 16.

5. The height of a cylindrical tank is equal to the diameter of a circular cross section; the volume of the tank is 1,000 gal What are its dimensions?

SUMMARY

Determination of the real roots of a polynomial equation with rational coefficients.

I. Try integral divisors of the constant term (Theorem I, page 27); if integral roots are found, remove the corresponding linear factors; *i.e.*, depress the equation.

Note. Be on the watch for upper and lower limits of the roots

II. If the leading coefficient of the last depressed equation is not unity when the remaining coefficients are integers, multiply its roots by a suitable factor k so as to obtain a new equation which satisfies the above conditions (Theorem III, page 27).

III. Find the integral roots of the last equation. The quotient of each of these numbers by k is a rational root of the given equation. Depress the last depressed equation each time a root is found.

IV. Make a table of values of the function $y = \varphi(x)$ where $\varphi(x) = 0$ is the last depressed equation, thus locating the irrational roots.

V. Compute the irrational roots by Horner's method.

In case of doubt as to whether all of the real roots have been found, find the maximum and minimum points on the graph.

CHAPTER III

RADICALS AND EQUATIONS CONTAINING RADICALS

In Chap. V of Vol. I it is shown that any radical can be replaced by a fractional exponent. As a preparation for the study of logarithms there was practice in multiplication of radicals, division of radicals, rationalizing the denominator, etc. In this chapter radicals will be added, subtracted, multiplied, and divided. Also, equations involving radicals will be solved.

16. Addition and subtraction of radicals. *Radicals having the same index and the same* **radicand*** *are added by adding the coefficients;* as, $3\sqrt{5} + 7\sqrt{5} - 4\sqrt{5} = 6\sqrt{5}$. If the indices are not the same and cannot be made the same by simplification or if the radicands are not the same and cannot be made the same by simplification, the sum must merely be indicated. For example, $3\sqrt{5} - 2\sqrt[3]{5}$ cannot be expressed as a monomial. The same is true of $3\sqrt{5} - 2\sqrt{7}$. Such an expression is called **a binomial surd.**

The next example has terms which appear to be unlike but which when simplified are alike, *i.e.*, have the same index and the same radicand.

$$2\sqrt{150} - 3\sqrt{54} + \sqrt{294} - \sqrt[4]{36} = 10\sqrt{6} - 9\sqrt{6} + 7\sqrt{6} - \sqrt{6} = 7\sqrt{6}.$$

EXERCISE XIII

Perform the indicated additions and subtractions when possible, simplifying radicals.

(*a*) $5\sqrt{2} - 3\sqrt{3} + 4\sqrt{12}$; (*b*) $\sqrt{98} + \sqrt{18} - 5$;

(*c*) $\sqrt{1000} - 8\sqrt{10} + \sqrt{90}$; (*d*) $\sqrt[3]{4394} + 5\sqrt[3]{2662}$;

(*e*) $30\sqrt{5} - 70\sqrt{45} + 10\sqrt{80}$; (*f*) $4\sqrt[3]{15} - 3\sqrt[6]{225}$;

(*g*) $2\sqrt{75} + 7\sqrt{27} - \sqrt{2}$; (*h*) $\sqrt{266805} - \sqrt{13520}$;

(*i*) $7\sqrt{a^3b^4c} - 3\sqrt{ac}$; (*j*) $\sqrt[3]{a^4b^2} + 5a\sqrt[3]{ab^2}$,

(*k*) $\sqrt{6a^2} + 3\sqrt{25b^2c}$; (*l*) $\sqrt{32a^2b} - \sqrt{18ab^2}$.

---

* The **radicand** is the expression under the radical sign.

17. Multiplication and division of radicals. Radicals having *the same index* may be multiplied and divided. Moreover, two radicals which do not have the same indices can be replaced by equivalent radicals having a common index and may then be multiplied or divided. *The* **product** *of two radicals with the same index is obtained by multiplying under the radical sign. The* **quotient** *of two radicals with the same index is obtained by dividing under the radical sign; e.g.,* $\sqrt{45}\cdot\sqrt{5} = \sqrt{225} = 15$, and $\sqrt{45} \div \sqrt{5} = \sqrt{9} = 3$.

The two statements of this section are proved by the use of the laws of exponents. Let us replace all radicals by fractional exponents. Then,

$$(1)\quad \sqrt{M}\cdot\sqrt{N} = M^{1/2}\cdot N^{1/2} = (M\cdot N)^{1/2} = \sqrt{MN}.$$

$$(2)\quad \sqrt{M}\cdot\sqrt[3]{N} = M^{1/2}\cdot N^{1/3} = M^{3/6}\cdot N^{2/6} = (M^3N^2)^{1/6} = \sqrt[6]{M^3N^2}.$$

$$(3)\quad \sqrt{M} \div \sqrt{N} = M^{1/2} \div N^{1/2} = \left(\frac{M}{N}\right)^{1/2} = \sqrt{\frac{M}{N}}.$$

While products and quotients of radicals are obtainable by multiplication and division under the radical sign, as a matter of experience the labor of computation is generally lessened if we use fractional exponents and *factor each radicand into powers of prime factors; e.g.,* $16 = 2^4$.

This makes the final simplification of the result easier.

ILLUSTRATIVE EXAMPLE. Find the product of

$$\sqrt[3]{18144a^2b} \text{ by } \sqrt{18144ab^4}.$$

SOLUTION.

$$\sqrt[3]{18144a^2b}\cdot\sqrt{18144ab^4} = (2^5\cdot 3^4\cdot 7a^2b)^{1/3}(2^5\cdot 3^4\cdot 7ab^4)^{1/2}$$

$$= 2\cdot 3(2^2\cdot 3\cdot 7a^2b)^{1/3}\cdot 2^2\cdot 3^2b^2(2\cdot 7a)^{1/2} = 2^3\cdot 3^3b^2(2^2\cdot 3\cdot 7a^2b)^{2/6}(2\cdot 7a)^{3/6}$$

$$= 216b^2(2^4\cdot 3^2\cdot 7^2\cdot a^4b^2)^{1/6}(2^3\cdot 7^3a^3)^{1/6} = 216b^2(2^7 3^2 7^5 a^7 b^2)^{1/6}$$

$$= 216\cdot 2ab^2\sqrt[6]{2\cdot 3^2\cdot 7^5ab^2} = 432ab^2\sqrt[6]{302526ab^2}.$$

In a problem such as this it would be very difficult to simplify the product if the given radicands had not been factored before the multiplication was performed.

For most purposes, and surely when any computation is to be made, the radical can be handled more easily if the denominator is put into a rational form.

EXERCISE XIV

Perform the indicated operations giving the results in simplified form and with no radicals in a denominator.

(*a*) $\sqrt{32} \cdot \sqrt{8}$; (*b*) $\sqrt[3]{32} \div 2\sqrt{8}$;
(*c*) $5\sqrt[3]{160} \div 2\sqrt[3]{25}$; (*d*) $4\sqrt{ab^2c^3} \div 2\sqrt{ab^3c}$;
(*e*) $6\sqrt{15} \cdot 4\sqrt[3]{75}$; (*f*) $6\sqrt[3]{a^4b^4c^8} \div \sqrt{8abc^3}$;
(*g*) $\dfrac{\sqrt{7} - \sqrt{14}}{2\sqrt{14} + 3\sqrt{7}}$; (*h*) $\dfrac{2\sqrt{3} - \sqrt{2}}{4\sqrt{3} + 2\sqrt{2}}$;
(*i*) $(2\sqrt{3} + 5)^2$; (*j*) $(1 + \sqrt{2})(1 - \sqrt{2})$.

18. Equations containing radicals. It is to be understood that when a term containing a radical appears in an equation the radical is not double or multiple valued but is to be taken with the sign which precedes it, or with the positive sign if there is no sign indicated. For example, in the equation $4 - \sqrt{x} = 7$, $x = 9$ is not a solution; in fact there is no solution. To solve such an equation we write $-\sqrt{x} = 3$, then square both sides, obtaining $x = 9$. However, before accepting $x = 9$ as a solution we substitute it in the original equation.

This is an illustration of a situation which is not infrequently met when equations containing radicals are solved; *viz.*, when it is necessary to square both members of the equation there is a possibility that one or more roots may be introduced. This means that the new equation may have solutions which are not solutions of the original equation; the two equations are not **equivalent** since **extraneous roots** have been introduced.

Note. Two equations are said to be equivalent if they have the same solutions. When an equation is multiplied by or divided by a constant which is not zero, an equivalent equation is obtained. The same is not true when an equation is multiplied by or divided by a function of the variable or variables.

A system of equations is said to be equivalent to a second system if the two systems have the same common solutions. Let $f(x,y) = 0$ and $F(x,y) = 0$ be two equations whose common solutions are desired. Equivalent systems are the following:

$$f(x,y) = 0,$$
$$af(x,y) + bF(x,y) = 0,$$

and

$$F(x,y) = 0,$$
$$af(x,y) + bF(x,y) = 0,$$

where a and b are numbers.

ILLUSTRATIVE EXAMPLE. Solve the following equation for x:

$$\sqrt{x^2 + 18} + 5x = 3.$$

SOLUTION. We write $\sqrt{x^2 + 18} = 3 - 5x$. After this we square both members and collect terms. This gives $24x^2 - 30x - 9 = 0$, from which $x = \frac{3}{2}$ and $x = -\frac{1}{4}$.

In this case, squaring did not introduce any extraneous roots as can be verified by substituting $x = \frac{3}{2}$ and $x = -\frac{1}{4}$ in the original equation.

EXERCISE XV

Solve the following equations for x:

(a) $10x = 39 + 2\sqrt{2x}$;

(b) $3x + \sqrt{3x - 4} = 1$;

(c) $3x + \sqrt{4 + 9x^2} = \dfrac{10}{\sqrt{4 + 9x^2}}$;

(d) $\sqrt{x - 9} = 1 + \sqrt{x}$;

(e) $\sqrt{10x + 6a} - \sqrt{3x} = \sqrt{x + 6a}$;

(f) $10w - 9\sqrt{2w} = 2$;

(g) $\sqrt{x} + \sqrt{3x + 2b} = \sqrt{8x + 2b}$;

(h) $l = \pi r\sqrt{r^2 + x^2}$.

SUMMARY

Radicals.
Operations on radicals.
Addition.
Subtraction.
Multiplication.
Division.
Rationalizing the denominator.
In equations.
Equivalent equations.
Extraneous roots.

The student should be able to describe the method of performing the above operations on radicals.

CHAPTER IV

SIMULTANEOUS LINEAR EQUATIONS IN MORE THAN TWO VARIABLES. DETERMINANTS

Two linear equations in two variables can be solved by the use of two-rowed determinants, *i.e.*, determinants of the second order.* Similarly, three linear equations in three variables can be solved by the use of **determinants of the third order,** four linear equations in four variables can be solved by the use of determinants of the fourth order, etc.

In this chapter we shall first solve three *general* linear equations in three unknowns. The solutions obtained will serve as a means of defining a determinant of the third order.

The properties of determinants will then be studied, determinants of order 3, and indeed of order n, after which it is a simple matter to apply the elementary theory of determinants to the solution of a system of n linear equations in n unknowns.

19. Simultaneous linear equations in three variables. We wish to find values (x, y, z) which satisfy the three linear equations

$$\begin{array}{l|c|c} a_1x + b_1y + c_1z = d_1, & c_2 & c_3 \\ a_2x + b_2y + c_2z = d_2, & -c_1 & \\ a_3x + b_3y + c_3z = d_3. & & -c_1 \end{array} \qquad (1)$$

The solution may be obtained in the following steps:

1. Eliminate z from the first and second equations, multiplying them, respectively, by c_2 and $-c_1$, as indicated to the right of the first vertical line, and afterward adding them.

2. Eliminate z from the first and third equations; the respective multipliers are c_3 and $-c_1$.

3. Eliminate y from the equations obtained in the first and second steps, using the respective multipliers $(b_1c_3 - b_3c_1)$ and $(b_2c_1 - b_1c_2)$. The result is $D \cdot x = D_1$ where the symbol D is defined as follows:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix};$$

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* Logsdon, "Elementary Mathematical Analysis," Vol. I, Chap. XI.

i.e.,

$$D = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1, \quad (2)$$

and where D_1 is the determinant obtained from D by replacing the coefficients of x, *viz.*, a_1, a_2, and a_3, by the constants d_1, d_2, and d_3, respectively, which in equations (1) constitute the right-hand members of the given system.

In a similar manner corresponding equations may be obtained for y and z; in these equations the symbols D_2 and D_3 are used to represent the determinants obtained from D by replacing the coefficients of y and of z, respectively, by the constants d_1, d_2, and d_3.

The equations which give the solutions of the system (1) are

$$\begin{aligned} D \cdot x &= D_1, \\ D \cdot y &= D_2, \\ D \cdot z &= D_3. \end{aligned} \quad (3)$$

If D is not zero each equation in (3) may be divided by D. The result is a unique solution, (x, y, z), of the given equations, which are therefore called **consistent.**

When D is zero there are several cases to be considered. This will be done on page 51 after the theory of determinants has been studied and a geometrical interpretation of a system of three linear equations in three unknowns has been made.

In equation (2) is given the definition of a determinant of order three. It is to be noted that when the elements a_1, b_1, etc., are numbers *a determinant is a* ***number*** expressed by a certain square array of the elements which combine in the manner indicated.

EXERCISE XVI

Find whether the following triples of equations are consistent and if so, find the common solution

(a) $2x - y + z = 0,$
$4x + 2y - 3z = 14,$
$2x + 3y + 5z = 13.$

(b) $3x - 2y - z = 0,$
$6x - 4y - 2z = 3,$
$9x - 6y - 3z = 2$

(c) $x + y - z = 4,$
$2x - 3y + 4z = -5,$
$4x - y + 2z = 3.$

(d) $3x + 2y - 3z = 6,$
$4x - 2y - 13z = -3,$
$-x + 4y + 4z = -3.$

20. Determinants. A three-rowed determinant whose elements are numbers is said to be **evaluated** when the number which is the sum of the six terms in the right member of equation

(2) has been found. The evaluation of a three-rowed determinant can be greatly simplified by the use of the theorems, definitions, and remarks which follow. The student should be able to supply all proofs which are omitted.

1. Any symbol, as a_2, c_1, etc., in D is called an **element** of the determinant.

2. If the elements in the first row and first column, for example, are deleted, the two-rowed determinant which remains is called the (first) **minor** of the element which stands in the first row and the first column. Thus each element has a minor. It is customary to denote the minor of a_1 by A_1, the minor of b_1 by B_1, etc. In the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

the minors of b_3 and of c_2 are, respectively,

$$B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \text{ and } C_2 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}.$$

3. The value of D as given in equation (2) may be expressed in terms of minors. It is

$$D = a_1A_1 - a_2A_2 + a_3A_3.$$

4. The value of D may equally well be obtained by multiplying the elements of *any column* or of *any row* by their respective minors and adding the products, provided that each element is associated with a plus sign or a minus sign (called the **positional sign**) according to the following scheme:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

For example, expanding according to the elements of the second row, we get

$$D = -a_2A_2 + b_2B_2 - c_2C_2.$$

Proof:

$$-a_2(b_1c_3 - b_3c_1) + b_2(a_1c_3 - a_3c_1) - c_2(a_1b_3 - a_3b_1) =$$
$$-a_2b_1c_3 + a_2b_3c_1 + a_1b_2c_3 - a_3b_2c_1 - a_1b_3c_2 + a_3b_1c_2.$$

These terms are identical with the terms in the definition of D given in equation (2).

The student should satisfy himself that the following six evaluations of D are correct.

By rows:

$$\begin{aligned} D &= a_1A_1 - b_1B_1 + c_1C_1, \\ D &= -a_2A_2 + b_2B_2 - c_2C_2, \\ D &= a_3A_3 - b_3B_3 + c_3C_3, \end{aligned}$$

By columns:

$$\begin{aligned} D &= a_1A_1 - a_2A_2 + a_3A_3, \\ D &= -b_1B_1 + b_2B_2 - b_3B_3, \\ D &= c_1C_1 - c_2C_2 + c_3C_3. \end{aligned}$$

5. The value of D is unchanged if the rows and columns are interchanged; *i.e.*,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

6. The value of $\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix}$ is mD.

7. When two *adjacent* rows or two adjacent columns are interchanged, the value of the determinant thus formed is $-D$. For example,

$$\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

8. The value of D is zero if each of the three elements in one row (or in one column) is zero.

9. The value of D is zero if the elements of two rows (or of two columns) are equal or are proportional. For example,

$$\begin{vmatrix} a_1 & ka_1 & c_1 \\ a_2 & ka_2 & c_2 \\ a_3 & ka_3 & c_3 \end{vmatrix} = c_1\begin{vmatrix} a_2 & ka_2 \\ a_3 & ka_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & ka_1 \\ a_3 & ka_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & ka_1 \\ a_2 & ka_2 \end{vmatrix} = 0$$

Let $k = 1$ in the last determinant and let us evaluate according to the elements of the first column. This gives an interesting corollary.

Cor. The number obtained by multiplying the elements of any row (or column) by the minors of any other row (or column) and adding, using the proper positional signs, is zero; in symbols,

$$a_1B_1 - a_2B_2 + a_3B_3 = 0,$$
$$a_1C_1 - a_2C_2 + a_3C_3 = 0, \text{ etc.}$$

10. The value of $\begin{vmatrix} a_1 & ma_1 + b_1 & c_1 \\ a_2 & ma_2 + b_2 & c_2 \\ a_3 & ma_3 + b_3 & c_3 \end{vmatrix}$ is D;

i.e., a determinant is unchanged if to the elements of a row (or column) are added the corresponding elements of any other row (or column) multiplied by the same number m.

Proof. Let us use the symbol D' for the determinant whose value we seek and let us expand according to the elements of the second column. Then

$$\begin{aligned} D' &= -(ma_1 + b_1)B_1 + (ma_2 + b_2)B_2 - (ma_3 + b_3)B_3 \\ &= -m(a_1B_1 - a_2B_2 + a_3B_3) + (-b_1B_1 + b_2B_2 - b_3B_3). \\ &\therefore D' = D. \end{aligned}$$

q.e.d.

ILLUSTRATIVE EXAMPLE. Evaluate the determinant $\begin{vmatrix} 2 & 1 & 3 \\ 1 & -4 & 6 \\ -3 & 2 & -2 \end{vmatrix}$.

SOLUTION. Let the elements of the second row be multiplied by -2 and added to the elements of the first row. This gives the numbers 0, 9, -9, respectively, as elements in the first row and leaves the second and third rows unchanged.

Now multiply the elements of the second row by 3 and add to the elements of the third row, leaving in the third row the numbers 0, -10, 16 and not changing the first and second rows. The determinant now is $\begin{vmatrix} 0 & 9 & -9 \\ 1 & -4 & 6 \\ 0 & -10 & 16 \end{vmatrix}$ which can be evaluated quite simply by expanding according to the elements of the first column. We get -1 times its minor $= -1 \cdot (9 \cdot 16 - 9 \cdot 10) = -54$.

EXERCISE XVII

1. Solve by determinants the problems in Exercise XVI in which D is not zero.

2. Evaluate the following determinants:

(a) $\begin{vmatrix} 2 & 5 & 1 \\ 0 & -3 & 4 \\ -5 & 7 & -2 \end{vmatrix}$ (b) $\begin{vmatrix} 7 & 4 & -1 \\ +3 & 5 & -7 \\ 8 & 3 & 2 \end{vmatrix}$

(c) $\begin{vmatrix} 4 & 2 & 4 \\ 6 & 3 & 6 \\ 5 & 1 & -5 \end{vmatrix}$ d) $\begin{vmatrix} -1 & 2 & 3 \\ -2 & 3 & 4 \\ -3 & 4 & 5 \end{vmatrix}$

(e) $\begin{vmatrix} 2 & 0 & 1 \\ 4 & -3 & 5 \\ 6 & 9 & 4 \end{vmatrix}$ (f) $\begin{vmatrix} 1 & 2 & 3 \\ 8 & -7 & 3 \\ 6 & -11 & -3 \end{vmatrix}$

(g) $\begin{vmatrix} 5 & 7 & 9 \\ 13 & 11 & 7 \\ 9 & 5 & 13 \end{vmatrix}$ (h) $\begin{vmatrix} 3 & -5 & 8 \\ -11 & 16 & 3 \\ 7 & 4 & 5 \end{vmatrix}$

3. For what value of x will the following determinants vanish?

(a) $\begin{vmatrix} 2 & -1 & 3 \\ 4x & 2 & -7 \\ 3 & 4 & -1 \end{vmatrix}$ (b) $\begin{vmatrix} x & -7 & 4 \\ 2 & 3 & -1 \\ 4x & 1 & 5 \end{vmatrix}$

(c) $\begin{vmatrix} 5x & -1 & 0 \\ 4 & x+1 & 3 \\ 1 & -2 & 1 \end{vmatrix}$ (d) $\begin{vmatrix} -1 & x & 3 \\ +3 & 6 & -9 \\ 1 & 2 & 4 \end{vmatrix}$

A determinant of order n where n is any positive integer not less than 2 is a symbol used to represent the sum of terms obtained by expanding according to the elements of any row or of any column. In this expansion each element of the row (or column) is multiplied by its minor which is a determinant of order $n-1$ and with each element is associated a positional sign. The scheme for positional signs has $+$ in the upper left-hand corner; thereafter the signs in both directions are alternately minus and plus.

The number of terms in the expansion of a determinant of order n is $n!$*

21. Rank of a determinant. *A determinant of order n whose value is not zero is said to be a determinant of* **rank** *n.* If the value of the determinant is zero and there is one (at least) minor determinant of order $n-1$ whose value is not zero, the rank of the determinant is said to be $n-1$.

If D and all minor determinants of order $n-1$ vanish, we examine the **minor determinants of order $n-2$**, *i.e.*, the determinants obtained by deleting two rows and two columns from D. If one (at least) of these determinants has a value which is not zero, the determinant is said to have rank $n-2$, etc.

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* The symbol $n!$ is read **n factorial** or **factorial n**. It indicates the product of all of the positive integers from 1 to n inclusive. Thus, $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$. (*Cf.* Vol. I, page 31.)

In the case where $n = 3$, let us consider again the three linear equations in three unknowns,

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3. \end{aligned} \tag{1}$$

A geometrical interpretation of these equations may be made as follows. In a space of three dimensions take as x-axis, y-axis, and z-axis three mutually perpendicular lines meeting at the origin of coordinates and call the plane containing the x-axis and y-axis, the xy-plane; the y-axis and z-axis, the yz-plane; the z-axis and x-axis, the xz-plane. A point $P(xyz)$ will be located by its distances from these three planes of reference; for example, x is the distance from the yz-plane measured on the x-scale, etc. When a point is given, there is for it only one set of coordinates (x, y, z) in this coordinate system; and conversely, for a given triple of numbers (x, y, z) there is one and only one point in the space whose coordinates are these numbers in the order agreed upon.

In such a system of coordinates, *the graph of a linear equation is a plane;* hence the possibilities of solutions of equations (1) are the same as the possibilities of intersection of three planes. The following tabulation shows how the geometric situations are related to the determinants which appear in equations (3).

THE PLANES	THE EQUATIONS	THE DETERMINANTS
The three planes meet in a point.	The equations are consistent.	The rank of D is 3.
One plane is parallel to the line of intersection of the two others.	The equations are inconsistent.	The rank of D is 2; at least one of D_1, D_2, D_3 has rank 3.
The three planes are parallel.	The equations are inconsistent	The rank of D is 1; D_1, D_2, D_3 have rank 2.
Two of the planes coincide and the third plane is parallel to them.	The equations are inconsistent.	The rank of D is 1; one of D_1, D_2, D_3 has rank 1, the others have rank 2.
The three planes have a line in common.	The equations are dependent.	The rank of D is 2; no one of D_1, D_2, D_3 has rank 3.
The three planes coincide.	The equations are dependent.	The rank of D is 1; no one of D_1, D_2, D_3 has rank greater than 1.

EXERCISE XVIII

1. Classify the following triples of equations geometrically. If they are consistent find the common point,

$$(a)\quad \begin{aligned} 2x - 3y + 7z &= -14,\\ 4x + y - 2z &= 14,\\ 2x - y + z &= 2; \end{aligned}$$

$$(b)\quad \begin{aligned} 3x + 5y - 8z &= -1,\\ 7x + 2y - 3z &= 17,\\ x + 6y - 9z &= -9; \end{aligned}$$

$$(c)\quad \begin{aligned} 5x + y &= 12,\\ 2x - y - 7z &= -12,\\ y + 4z &= 10; \end{aligned}$$

$$(d)\quad \begin{aligned} x + 4y - 4z &= 15,\\ 3x + y + 5z &= 7,\\ 4x - 3y + z &= -18; \end{aligned}$$

$$(e)\quad \begin{aligned} x + y - z &= 1,\\ x + y - z &= 3,\\ x - y + z &= 4; \end{aligned}$$

$$(f)\quad \begin{aligned} x + y - z &= 1,\\ 2x - 3y - z &= -4,\\ y + z &= -6; \end{aligned}$$

$$(g)\quad \begin{aligned} x + y - z &= 1,\\ 2x - 3y - z &= -4,\\ y + z &= 0; \end{aligned}$$

$$(h)\quad \begin{aligned} x + z &= 1,\\ y + z &= 1,\\ x - y + 3z &= -1. \end{aligned}$$

2. Multiply the three equations (1) on page 51 respectively, by A_1, A_2, A_3, add the resulting equations, and obtain the first formula in equation (3); *viz.*, $D \cdot x = D_1$. In the same way use B_1, B_2, B_3 as the respective multipliers and obtain $D \cdot y = D_2$.

22. Simultaneous linear equations in *n* variables. By the method of elimination used in Sec. 19, and also by the method in Problem 2 in the last set of exercises, it can be shown that the variables $x_1, x_2, \ldots, x_n$ appearing in a system of n linear equations satisfy the following relations:

$$D \cdot x_1 = D_1, \quad D \cdot x_2 = D_2, \quad \cdots, D \cdot x_n = D_n. \tag{4}$$

If D is not zero, equations (4) furnish a unique set of numbers $(x_1, x_2, \ldots, x_n)$ which satisfy each of the equations of the given system; *the equations are consistent.*

When D is zero, the number of solutions depends on the ranks of D, D_1, D_2, . . . D_n. Such systems are discussed in texts on the Theory of Equations.

23. Some applications of determinants. A number of formulas of analytical geometry can be expressed by the use of determinants. We shall prove first that *the area of a triangle whose vertices have the coordinates* (x_1, y_1), (x_2, y_2), *and* (x_3, y_3) *is given by*

$$S = \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1 \end{vmatrix}. \tag{5}$$

Proof. The area of the triangle $P_1P_2P_3$ is equal to the sum of the areas of the two trapezoids $M_3M_2P_2P_3$ and $M_2M_1P_1P_2$ (Fig. 17) diminished by the area of the trapezoid $M_3M_1P_1P_3$; *i.e.*,

$$S = \tfrac{1}{2}(y_2 + y_3)(x_2 - x_3) + \tfrac{1}{2}(y_1 + y_2)(x_1 - x_2) - \tfrac{1}{2}(y_1 + y_3)(x_1 - x_3),$$
$$S = \tfrac{1}{2}(x_2y_3 - x_3y_2 - x_1y_3 + x_3y_1 + x_1y_2 - x_2y_1).$$

But the right member of the last equation is equal to the right member of equation (5) as is easily seen by developing the determinant according to the elements of the last column.

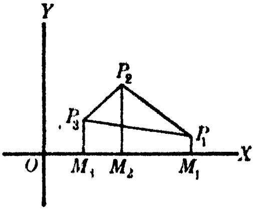

FIG. 17.

Note. The *rows* in the determinant in (5) contain the points $P_1P_2P_3$ as they are reached when the perimeter of the triangle is traversed in the *positive direction* starting at P_1. Let the rows be arranged in the sequence $x_1 \;\; y_1 \;\; 1$, $x_3 \;\; y_3 \;\; 1$, $x_2 \;\; y_2 \;\; 1$, which corresponds to the order in which the points are reached when the perimeter of the triangle is traversed in the *negative direction* starting at P_1. By this rearrangement of rows the sign of the determinant is changed. We conclude that *a positive number is obtained for the area of the triangle in Fig. 17 if the vertices $P_1P_2P_3$ are in counterclockwise order.*

In Fig. 17, let P_1 and P_2 be fixed points and consider a point P which takes various positions in the plane. The area of the triangle PP_1P_2 may be written out in determinantal form for any position of P. In particular, if P takes a position on the line P_1P_2 the area of the triangle PP_1P_2 is zero; *i.e.*, for P on the line determined by P_1 and P_2 we have

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \tag{6}$$

Equation (6) is the equation of the line P_1P_2, since it is satisfied by the coordinates of any point on the line and is not satisfied by the coordinates of any point not on the line.

DEFINITIONS. Three or more points are said to be **collinear** if they lie in a straight line; two or more lines which meet at a point A are said to **concur** at A; three or more lines are said to be **concurrent** if they pass through the same point.

From equation (6) we conclude that three points (x_3, y_3), (x_1, y_1), and (x_2, y_2) are concurrent if the determinant

$$\begin{vmatrix} x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$$

vanishes.

The condition that three lines be concurrent is expressible as a determinant as follows: Let the equations of the three lines L_1, L_2, and L_3 be

$$\begin{aligned} L_1&: \quad a_1x + b_1y + c_1 = 0,\\ L_2&: \quad a_2x + b_2y + c_2 = 0,\\ L_3&: \quad a_3x + b_3y + c_3 = 0. \end{aligned} \tag{7}$$

If L_2 and L_3 meet in a point, its coordinates are expressible in terms of the determinants

$$\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}, \qquad \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \qquad \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

obtained from the coefficients appearing in the equations of L_2 and L_3. These determinants are the minors of the elements c_1, b_1, and a_1, respectively, in the determinant of the third order obtained from the equations in (7) by erasing everything except the a's, b's, and c's. Hence we can express the coordinates (x', y') of the point of intersection of L_2 and L_3 as follows:

$$x' = \frac{A_1}{C_1}, \qquad y' = \frac{-B_1}{C_1}, \tag{8}$$

where as usual we use A_1 to denote the minor of the element a_1 in the determinant D on page 45, and where C_1 is supposed not zero.

The three lines are concurrent if the coordinates of the point (x', y') satisfy the equation of L_1, *i.e.*, if

$$\frac{a_1A_1}{C_1} - \frac{b_1B_1}{C_1} + c_1 = 0.$$

This equation when multiplied by C_1 becomes the equivalent equation

$$a_1A_1 - b_1B_1 + c_1C_1 = 0. \tag{9}$$

Now the left member of equation (9) is an expression for the value of the determinant whose elements are the coefficients in equations (7); hence, *the three lines whose equations are* $a_ix + b_iy + c_i = 0$ $(i = 1, 2, 3)$ *are concurrent if and only if the determinant of the coefficients is zero., i.e. if*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{10}$$

EXERCISE XIX

1. Compute the areas of the triangles whose vertices are as follows:

(*a*) $(1, -2)$, $(3, 1)$, $(0, 4)$; (*b*) $(0, 0)$, $(1, 1)$, $(-2, 1)$;
(*c*) $(-4, -4)$, $(0, 0)$, $(2, -2)$; (*d*) $(1, 2)$, $(-3, 1)$, $(4, 0)$;
(*e*) $(4, 2)$, $(2, 1)$, $(-1, 3)$.

2. Write in determinantal form the equations of the lines determined by the following pairs of points:

(*a*) $(1, -3)$, $(2, 4)$; (*b*) $(-3, 1)$, $(4, 2)$;
(*c*) $(0, 0)$, $(-2, 5)$; (*d*) $(2, 0)$, $(0, -5)$;
(*e*) $(-2, 7)$, $(1, -3)$.

3. Which of the following triples of points are collinear?

(*a*) $(4, 2)$, $(-1, 2)$, $(10, 2)$; (*b*) $(0, 0)$, $(6, -3)$, $(-2, 1)$;
(*c*) $(1, 1)$, $(-3, 2)$, $(4, 1)$; (*d*) $(1, -3)$, $(2, 5)$, $(3, 13)$;
(*e*) $(4, 0)$, $(0, -3)$, $(-2, 1)$.

4. Which of the following triples of lines are concurrent?

(*a*)
$$\begin{aligned} 4x - y + 3 &= 0, \\ 2x + 3y - 1 &= 0, \\ 2x - 4y + 5 &= 0; \end{aligned}$$

(*b*)
$$\begin{aligned} x + y - 3 &= 0, \\ 5x + 7y + 10 &= 0, \\ 2x + 4y + 19 &= 0; \end{aligned}$$

(*c*)
$$\begin{aligned} x - 2y + 1 &= 0, \\ 2x + y - 3 &= 0, \\ 5x - y - 4 &= 0; \end{aligned}$$

(*d*)
$$\begin{aligned} 3x + y - 2 &= 0, \\ 4x + y + 3 &= 0, \\ 10x + 3y + 4 &= 0 \end{aligned}$$

SUMMARY

- Determinants.
  - Element.
  - Minor.
  - Order.
  - Rank.
  - Properties.
- Simultaneous linear equations.
  - Solution by formula.
  - Geometrical interpretation for $n = 3$.
- Some applications of determinants.
  - Area of a triangle determined by three points.
  - Equation of the line through two points.
  - Condition that three points be collinear.
  - Condition that three lines be concurrent.

CHAPTER V

TRIGONOMETRIC ANALYSIS AND APPLICATIONS

The four trigonometric functions already studied—sine, cosine, tangent, and cotangent—are sufficient for problems of computation but, for purposes of analysis and in connection with various notions of the calculus, it is convenient to give names to the reciprocals of the sine and cosine. These two functions, called, respectively, the **cosecant** and **secant** functions, will be studied in this chapter; also extensive practice is given in the manipulation of formulas under the captions of trigonometric identities and the solution of trigonometric equations. These topics are followed by some applications to the triangle, in particular the derivation of area formulas.

24. The secant and cosecant functions. By definition

$$\sec \theta = \frac{1}{\cos \theta} = \frac{r}{x}; \tag{1}$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{r}{y}. \tag{2}$$

The graph of the secant function, $y = \sec x$, can be constructed most easily by sketching first the graph of the cosine function and using the following properties of reciprocals:

a. The reciprocal of $+1$ is $+1$; that of -1 is -1.

b. The reciprocal of a positive (negative) number is a positive (negative) number.

c. The reciprocal of a number numerically less than unity is numerically greater than unity.

We conclude,

a. When $\cos x = +1$ or -1 the same is true of $\sec x$.

b. When $\cos x$ is positive the same is true of $\sec x$.

c. No ordinate of $y = \sec x$ can be less than one unit in length.

Moreover, when $\cos x$ approaches zero, $\sec x$ becomes *large without bound.* In other words *the zeros of* $\cos x$ *are the poles of* $\sec x$:

$$\cos\,(2m+1)\frac{\pi}{2} = 0, \qquad (m = \text{integer})$$

$$\sec\,(2m+1)\frac{\pi}{2} = \infty.$$

The period of the secant is 360° = 2π radians.

The graph is given in Fig. 18, where the cosine curve is drawn very lightly, the asymptotes are dotted, and the heavy curve is $y = \sec x$.

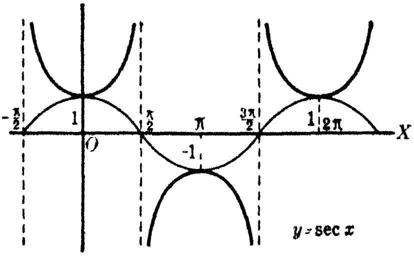

FIG. 18

The graph of $y = \csc x$ may be drawn by tracing around the pattern of the secant curve after it has been moved 90° to the right on the x-axis. *The zeros of* $\sin x$ *are the poles of* $\csc x$; *i.e.*,

$$\csc m\pi = \infty. \qquad (m = \text{integer})$$

EXERCISE XX

1. Draw the graph of the cosecant function.
2. Write the values of $\sec x$ and $\csc x$ for x = 0°, 30°, 45°, 60°, 90°, 120°, 135°, 210°, 225°, 240°, 330°, 570°
3. Compute the remaining functions of θ if $\theta = \sec^{-1} 3$
4. Compute the remaining functions of θ if $\theta = \csc^{-1}(-2)$.
5. Draw the graphs of $y = \text{arc sec } x$ and $y = \text{arc csc } x$.
6. Draw the graph of $y = \sec 2x$; $y = \frac{1}{4} \csc 3x$.
7. Draw the graph of $y = \sec x + \csc x$

25. Derivatives. To compute the derivative of $\sec x = 1/\cos x$ we may apply the power formula $D_x\frac{1}{u} = -\frac{1}{u^2}D_x u$. This gives

$$D_x \sec x = D_x \frac{1}{\cos x} = -\frac{1}{\cos^2 x}D_x \cos x = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x};$$

and finally,

$$\mathbf{D_x \sec x = \sec x \tan x.} \tag{3}$$

Similarly,

$$\mathbf{D_x \csc x = -\csc x \cot x.} \tag{4}$$

The graph in Fig. 18 shows that the secant curve has maximum points at $\pm\pi$, $\pm 3\pi$, etc., and minimum points at 0, $\pm 2\pi$, $\pm 4\pi$, etc. This should be evident from the derivative of sec x. The derivative, sec x tan x, can be zero only when tan $x = 0$ (since sec x is never zero), *i.e.*, at $x = m\pi$. Hence the even multiples of π, at which sec $x = +1$, are minima and the odd multiples of π, at which sec $x = -1$, are maxima.

Similarly the derivative in (4) shows that the maxima and minima of the graph of the cosecant function occur at the zeros of cot x, *viz.*, at odd multiples of $\pi/2$.

There are two additional trigonometric functions which are sometimes used for algebraic simplification of certain formulas. They are the **versed sine** and the **coversed sine** and are defined as follows:

$$\begin{aligned} \operatorname{vers} x &= 1 - \cos x, \\ \operatorname{covers} x &= 1 - \sin x. \end{aligned} \tag{5}$$

From the definitions, we get easily

$$\begin{aligned} D_x(\operatorname{vers} x) &= \sin x, \\ D_x(\operatorname{covers} x) &= -\cos x. \end{aligned} \tag{6}$$

Derivatives of the inverse trigonometric functions. From the derivatives of the direct trigonometric functions we can obtain the derivatives of the inverse functions. For example, to obtain the derivative of

$$y = \text{arc} \sin x,$$

we write

$$\sin y = x$$

and differentiate both members of the last equation with respect to x. This gives

$$\begin{aligned} \cos y \, D_x y &= 1, \\ D_x y &= \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}. \end{aligned}$$

More generally, if u is a function of x, we write

$$D_x(\text{arc} \sin u) = \frac{1}{\sqrt{1 - u^2}} D_x u. \tag{7}$$

From the equation preceding (7) we infer that the derivative of $y = \text{arc} \sin x$ is double valued, since for a given $x = x_0$ ($|x_0| < 1$) there are two values of $\sqrt{1 - x_0^2}$. This is indeed true.

The arc sine function has an infinite number of real values for any admissible x_0. These values are the ordinates of the points in Fig. 19 where the graph is cut by the line $x = x_0$. At half of these points the slope is positive and is equal to $+1/\sqrt{1 - x_0^2}$; at the remaining points the slope is negative and is equal to $-1/\sqrt{1 - x_0^2}$. Inspection will show that for the arc indicated by *AOB* the slope is positive; *i.e.*, the derivative is correctly given by use of the positive sign with the radical. This part of the graph is called the **principal value of the arc sine function** and it is agreed that we shall give $+1/\sqrt{1 - x^2}$ as the derivative of arc sin x with the understanding that the formula is correct for the principal value of the arc sine and that if it be desired to find the slope of the tangent to $y =$ arc sin x at any point not on the arc *AOB* it can be found by reference to the graph.

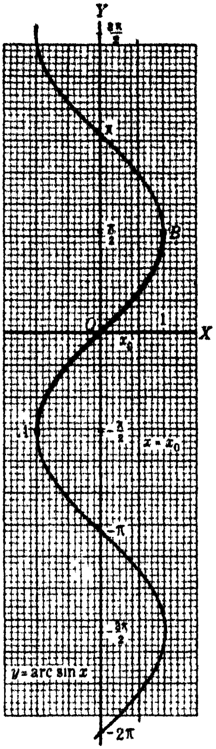

FIG. 19.

The following table shows the derivatives of the inverse functions. When the radicals are taken with the signs indicated in the table, the derivatives are unambiguous as regards sign for the principal values indicated.

Function	Derivative	Principal value
$y =$ arc sin x	$\frac{1}{\sqrt{1 - x^2}}$	$-\frac{\pi}{2} \leqq y \leqq \frac{\pi}{2}$
$y =$ arc cos x	$\frac{-1}{\sqrt{1 - x^2}}$	$0 \leqq y \leqq \pi$
$y =$ arc tan x	$\frac{1}{1 + x^2}$	$-\frac{\pi}{2} \leqq y \leqq \frac{\pi}{2}$
$y =$ arc cot x	$\frac{-1}{1 + x^2}$	$0 \leqq y \leqq \pi$
$y =$ arc sec x	$\frac{1}{x\sqrt{x^2 - 1}}$	$0 \leqq y \leqq \pi$
$y =$ arc csc x	$\frac{-1}{x\sqrt{x^2 - 1}}$	$-\frac{\pi}{2} \leqq y \leqq \frac{\pi}{2}$

The student should sketch the graphs of the inverse trigonometric functions* and keep clearly in mind which portions of the curves have been designated as principal values.

EXERCISE XXI

1. Verify the following derivatives.

(a) $D_x\left(\text{arc sin }\frac{u}{a}\right) = \frac{1}{\sqrt{a^2 - u^2}}D_x u$; (b) $D_x\left(\text{arc cos }\frac{u}{a}\right) = \frac{-1}{\sqrt{a^2 - u^2}}D_x u$;

(c) $D_x\left(\text{arc tan }\frac{u}{a}\right) = \frac{a}{a^2 + u^2}D_x u$; (d) $D_x\left(\text{arc cot }\frac{u}{a}\right) = \frac{-a}{a^2 + u^2}D_x u$;

(e) $D_x\left(\text{arc sec }\frac{u}{a}\right) = \frac{a}{u\sqrt{u^2 - a^2}}D_x u$; (f) $D_x\left(\text{arc csc }\frac{u}{a}\right) = \frac{-a}{u\sqrt{u^2 - a^2}}D_x u$;

(g) $D_x\left(\text{arc vers }\frac{u}{a}\right) = \frac{1}{\sqrt{2au - u^2}}D_x u$; (h) $D_x\left(\text{arc covers }\frac{u}{a}\right) = \frac{-1}{\sqrt{2au - u^2}}D_x u$.

2. Find the derivatives of the following functions:

(a) $y = \sec^2(2x - 1)$. *Ans.* $D_x y = 4\sec^2(2x - 1)\tan(2x - 1)$.

(b) $y = \text{arc sec } 3x^2$. *Ans.* $D_x y = \frac{2}{x\sqrt{9x^4 - 1}}$.

(c) $y = 3\tan x \cos 3x$. *Ans.* $-9\tan x \sin 3x + 3\cos 3x \sec^2 x$.

(d) $y = \frac{1}{4}\text{arc tan }\frac{x^2}{\sqrt{2}}$. *Ans.* $\frac{x}{\sqrt{2}[2 + x^4]}$.

(e) $y = 4\sin^3 4x$; (f) $y = \sec x^2 \sin x^2$;

(g) $y = 4\cot 3x$; (h) $y = a\tan^3 bx$

3. Find the maxima and minima of the following functions.

(a) $y = \cos 2x$; (b) $y = \tan(3x + 1)$;

(c) $y = \sin x + \cos x$; (d) $y = \sin^2 x$;

(e) $y = \text{arc cot } 3x$; (f) $y = \tan^2 3x$.

4. Show that $\sin(\text{vers } x) = \sqrt{2ax - x^2}$.

5. Prove the following relations:

(a) $\sin(\tan^{-1} a) = \pm\frac{a}{\sqrt{1 + a^2}}$; (b) $\cos(\sec^{-1} a) = \frac{1}{a}$.

26. Fundamental relations. From the definitions in terms of x, y, and r of the six trigonometric functions and the relation $x^2 + y^2 = r^2$ which connects the coordinates of a point P on the terminal side of the angle θ, we can write down the following **fundamental relations** of the functions themselves:

(1) $\sin^2\theta + \cos^2\theta = 1$, (2) $\sec^2\theta = 1 + \tan^2\theta$,

(3) $\csc^2\theta = 1 + \cot^2\theta$, (4) $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{\cot\theta}$,

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* *Cf.* Vol. I, p. 48

$$(5)\ \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}, \qquad (6)\ \sec\theta = \frac{1}{\cos\theta}, \tag{8}$$

$$(7)\ \csc\theta = \frac{1}{\sin\theta}, \qquad (8)\ \operatorname{vers}\theta = 1 - \cos\theta,$$

$$(9)\ \operatorname{covers}\theta = 1 - \sin\theta.$$

The student should prove relations (2) and (3). Equations (8) may be used to express the seven remaining functions in terms of any one of them. This can not be done rationally because of the three quadratic equations (1), (2), and (3) in (8). An example will illustrate.

Illustrative Example Express the remaining functions in terms of sin x

Solution. For convenience we use s to designate sin x In the results below, the number (7) for example indicates that the result was obtained by use of equation (7) in the set of equations labeled (8) The order in which the results were obtained is (7), (1), (6), etc. We have

$$(7)\ \csc x = \frac{1}{s}, \qquad (1)\ \cos x = \pm\sqrt{1 - s^2}, \quad (6)\ \sec x = \frac{1}{\pm\sqrt{1 - s^2}},$$

$$(4)\ \tan x = \frac{s}{\pm\sqrt{1 - s^2}}, \quad (5)\ \cot x = \frac{\pm\sqrt{1 - s^2}}{s}, \quad (9)\ \operatorname{covers} x = 1 - s,$$

$$(8)\ \operatorname{vers} x = 1 \mp \sqrt{1 - s^2}$$

While it is not possible to express the remaining functions *rationally* in terms of one of them, *each trigonometric function can be expressed rationally in terms of sine and cosine together.* This is of considerable importance in the solution of trigonometric equations, in reducing identities, and in integrating trigonometric functions.

Illustrative Example. Reduce to an expression involving only sines and cosines and simplify

$$\frac{\cos x}{1 - \tan x} + \frac{\sin x}{1 - \cot x}.$$

Solution. We have

$$\frac{\cos x}{1 - \dfrac{\sin x}{\cos x}} + \frac{\sin x}{1 - \dfrac{\cos x}{\sin x}}$$

$$= \frac{\cos^2 x}{\cos x - \sin x} + \frac{\sin^2 x}{\sin x - \cos x}$$

$$= \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x} = \cos x + \sin x.$$

EXERCISE XXII

1. Express the remaining functions in terms of

(*a*) $\cos x$; (*b*) $\tan x$; (*c*) $\sec x$; (*d*) $\csc x$.

2. Reduce each of the following to a rational expression involving only sines and cosines:

(*a*) $2\tan x \sec x - 3\csc x - \sin x$; (*b*) $\tan x \tan y (\cot x + \cot y)$;

(*c*) $\dfrac{\sec\theta + \csc\theta}{\sec\theta - \csc\theta}$; (*d*) $\dfrac{\tan x + \sec x - 1}{\tan x - \sec x + 1}$;

(*e*) $\dfrac{\sec x + \csc x}{\tan x + \cot x}$; (*f*) $\dfrac{\cos x \cot x - \sin x \tan x}{\csc x - \sec x}$;

(*g*) $\cot x + \dfrac{\tan x}{1 + \sec x}$; (*h*) $2 \operatorname{vers} x - \operatorname{vers}^2 x$.

27. Trigonometric identities. An equation containing trigonometric functions may be an identity; *i.e.*, the two members of the equation may be merely two ways of writing the same thing. For example, consider the last illustrative example. It shows that

$$\frac{\cos x}{1 - \tan x} + \frac{\sin x}{1 - \cot x} = \cos x + \sin x.$$

From the way in which the right-hand member was obtained from the left-hand member, it is evident that this is indeed an identical equation and is, consequently, true for all values of the variable x. *To prove an identity* let us agree to take the left-hand member of the given equation and change its form (generally by use of equations (8)) until we have put it into the form of the right-hand member.

ILLUSTRATIVE EXAMPLE 1. Prove the identity $\dfrac{1 + \sin x}{\cos x} = \dfrac{\cos x}{1 - \sin x}$.

SOLUTION.

$$\frac{1 + \sin x}{\cos x} = \frac{(1 + \sin x)(1 - \sin x)}{\cos x(1 - \sin x)} = \frac{1 - \sin^2 x}{\cos x(1 - \sin x)} = \frac{\cos^2 x}{\cos x(1 - \sin x)}$$

$$= \frac{\cos x}{1 - \sin x}. \qquad \textit{q.e.d.}$$

ILLUSTRATIVE EXAMPLE 2. Prove the identity $\dfrac{\sec^2\left(\frac{\pi}{4} + \frac{1}{2}x\right)}{2\tan\left(\frac{\pi}{4} + \frac{1}{2}x\right)} = \sec x.$

SOLUTION. We can change the form of the left member as follows:

$$\frac{\sec^2\left(\frac{\pi}{4} + \frac{1}{2}x\right)}{2\tan\left(\frac{\pi}{4} + \frac{1}{2}x\right)} = \frac{1}{2\sin\left(\frac{\pi}{4} + \frac{1}{2}x\right)\cos\left(\frac{\pi}{4} + \frac{1}{2}x\right)} = \frac{1}{\sin\left(\frac{\pi}{2} + x\right)} =$$

$$\csc\left(\frac{\pi}{2} + x\right) = \sec x. \quad \textit{q.e.d.}$$

28. Trigonometric equations. A trigonometric equation which is not an identical equation expresses a relationship between the functions of an angle which is true for certain but not for all angles. *To solve a trigonometric equation* means to find all of the values of the variable which render the two members of the equation equal.

ILLUSTRATIVE EXAMPLE 1. Solve the equation $\sin^2\theta + 2 = 2\cos\theta$.

SOLUTION. We replace $\sin^2\theta$ by $1 - \cos^2\theta$, obtaining a quadratic equation in $\cos\theta$ which may be factored at sight. Thus

$$\begin{aligned} 1 - \cos^2\theta + 2 &= 2\cos\theta, \\ \cos^2\theta + 2\cos\theta - 3 &= 0, \\ (\cos\theta - 1)(\cos\theta + 3) &= 0. \end{aligned}$$

The product of two factors is zero if either of the factors is zero; hence the given equation is satisfied if $\cos\theta = 1$ and if $\cos\theta = -3$. From $\cos\theta = 1$, we have $\theta = 2m\pi$ (m, an integer). These are the only real solutions.

ILLUSTRATIVE EXAMPLE 2. Solve the equation $4\sin\theta + 3\cos\theta = 3$.

SOLUTION. Let $\cos\theta$ be replaced by $\sqrt{1 - \sin^2\theta}$ and let the two members (after rearrangement) be squared to remove radicals. The solution is as follows:

$$\begin{aligned} 4\sin\theta + 3\sqrt{1 - \sin^2\theta} &= 3, \\ 3\sqrt{1 - \sin^2\theta} &= 3 - 4\sin\theta, \\ 9 - 9\sin^2\theta &= 9 - 24\sin\theta + 16\sin^2\theta, \\ 25\sin^2\theta - 24\sin\theta &= 0, \\ \sin\theta(25\sin\theta - 24) &= 0. \end{aligned}$$

The last equation is satisfied if either factor is zero. Hence we have first,

$$\sin\theta = 0, \therefore \theta = m\pi;$$

second,

$$\sin\theta = {}^{24}\!/_{25}, \therefore \theta = 73^\circ.74 + m.360^\circ \text{ and } \theta = 106^\circ.26 + m.360^\circ.$$

It cannot, however, be assumed that every solution of the equation obtained by squaring is also a solution of the given equation. (*Cf.* page 43.) Let us test the two solutions in turn:

First, when $\sin\theta = 0$, $\cos\theta = 1$ or -1 according as m in $m\pi$ is even or odd. The equation $4\sin\theta + 3\cos\theta = 3$ is satisfied for $\sin\theta = 0$, $\cos\theta = +1$ but not for $\sin\theta = 0$, $\cos\theta = -1$.

Second, when $\sin\theta = {}^{24}\!/_{25}$; $\cos\theta = {}^{7}\!/_{25}$ or $-{}^{7}\!/_{25}$. The given equation is satisfied by $\sin\theta = {}^{24}\!/_{25}$, $\cos\theta = -{}^{7}\!/_{25}$ but not by $\sin\theta = {}^{24}\!/_{25}$, $\cos\theta = {}^{7}\!/_{25}$. Hence the solutions of the given equation are

$$\theta = 2m\pi, \text{ and } \theta = 106^\circ.26 + m.360^\circ.$$

If functions other than sine and cosine appear in a trigonometric equation, the student should examine it to see whether the solution may be obtained easily in terms of one of those functions, but if nothing simpler presents itself, an equivalent equation can always be found which contains one or both of $\sin\theta$ and $\cos\theta$.

This equation may be rational in one of these functions as in the first illustrative example or it may be necessary to square it one or more times before an equation is obtained which is rational in one function only. In the latter case, each solution of the final equation must be carefully tested in the original equation.

EXERCISE XXIII

1. Prove the following trigonometric identities:

(*a*) $\dfrac{\tan\theta}{\sin\theta} = \sec\theta$;

(*b*) $\dfrac{\sin^2 x}{\tan^2 x} = 1 - \sin^2 x$;

(*c*) $\cos^2 x(1 + \tan^2 x) = 1$;

(*d*) $\tan\,(\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3) = 0$;

(*e*) $\dfrac{1 + \cos 2A}{1 - \cos 2A} = \cot^2 A$;

(*f*) $\dfrac{1 + \tan^2 x}{1 - \tan^2 x} = \sec 2x$;

(*g*) $\tan^{-1}\dfrac{x}{\sqrt{1 - x^2}} = \sin^{-1} x$;

(*h*) $\dfrac{1 - \sin 2x}{\cos 2x} = \dfrac{1 - \tan x}{1 + \tan x}$;

(*i*) $\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x$;

(*j*) $\sec^4 x - 1 = \tan^4 x + 2\tan^2 x$;

(*k*) $\cos A + \cos 3A + \cos 5A + \cos 7A = 4\cos A \cos 2A \cos 4A$;

(*l*) $\cos\,(A + 6B) - \cos\,(A - 6B) = -2\sin A \sin 6B$;

(*m*) $(1 + \tan\theta)(1 + \cot\theta)\sin\theta\cos\theta = 1 + 2\sin\theta\cos\theta$;

(*n*) $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$;

(*o*) $\sin\,(\sin^{-1} x + \sin^{-1} y) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$.

2. Solve the following trigonometric equations:

(*a*) $1 + \tan^2\theta = 4\cos^2\theta$;

(*b*) $4\sec^2 x - 7\tan^2 x = 3$;

(*c*) $\sin x - \cos x = 0$;

(*d*) $\tan A = \sin A$;

(*e*) $2\cos y + 3\sec y = 5$;

(*f*) $\sin x = \cos 2x$;

(*g*) $3\cos x + 5\sin x = 4$;

(*h*) $\sin^2 x - \cos^2 x = 0$;

(*i*) $\sin^2 x \sec x \csc x \cos x = \frac{1}{3}$;

(*j*) $\cos x \cos 3x = -\frac{1}{2}$;

(*k*) $\tan 2x + 2\sin x = 0$;

(*l*) $\sec^2 x + 3\tan^2 x = 5$;

(*m*) $2\sin^{-1} x = 3\cos^{-1} x$;

(*n*) $\cot^{-1} x + \cot^{-1} 2x = 3\pi/4$.

29. Area formulas. When parts which determine a triangle are given* we can find a formula for the area of the triangle which contains the symbols used to represent the given parts. Other interesting area formulas involve the radius of the inscribed circle, the radius of the circumscribed circle, etc. From the formula obtained in elementary geometry, *viz.*,

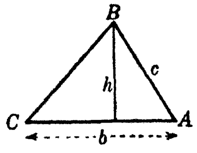

FIG. 20.

$$S = \tfrac{1}{2}bh, \tag{9}$$

where S = area of the triangle whose base is b and whose altitude is h, we obtain and tabulate the various cases, recalling that, when a formula is obtained, two corresponding formulas are derived from it by a cyclic interchange on the letters. This is equivalent to a change of lettering in Fig. 20 in which A is replaced by B, B by C, C by A, with the corresponding replacements on a, b, and c.

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* *Cf.* Vol. I, pp. 88 *ff.*

I. *Given two sides and the included angle.* In equation (9) let h be replaced by $a \sin C$; the resulting equation in its three forms is

$$\begin{aligned} S &= \tfrac{1}{2}ab \sin C, \\ S &= \tfrac{1}{2}bc \sin A, \\ S &= \tfrac{1}{2}ca \sin B. \end{aligned} \tag{10}$$

II. *Given two angles and the included side.* In the first equation in (10) let b be replaced by $\frac{a \sin B}{\sin A}$ obtained from the law of sines; the resulting equation in its three forms is

$$\begin{aligned} S &= \frac{1}{2}a^2\frac{\sin B \sin C}{\sin A}, \\ S &= \frac{1}{2}b^2\frac{\sin C \sin A}{\sin B}, \\ S &= \frac{1}{2}c^2\frac{\sin A \sin B}{\sin C}. \end{aligned} \tag{11}$$

III. *Given the three sides.* The formula desired is obtained from certain geometric relations of the parts of Fig. 21 plus purely algebraic processes starting with the half-angle formulas for sine and cosine (Vol. I, page 102) and using the law of cosines (Vol. I, page 86). A new symbol s is introduced. It is used to represent *one-half of the perimeter of the triangle; i.e.*, $2s = a + b + c$, whence $2(s - a) = b + c - a$, $2(s - b) = a - b + c$, and $2(s - c) = a + b - c$. Before the formula for the area of the triangle in terms of a, b, c, and s is obtained, formulas in these symbols for the functions of half an angle and for the radius of the inscribed circle are derived. The student will be able without explanation to follow the steps in the development.

$$2 \sin^2 \frac{A}{2} = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - (b - c)^2}{2bc} = \frac{(a + b - c)(a - b + c)}{2bc} = \frac{2(s - c)(s - b)}{bc}.$$

$$2 \cos^2 \frac{A}{2} = 1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{(b + c)^2 - a^2}{2bc} = \frac{(a + b + c)(b + c - a)}{2bc} = \frac{2s(s - a)}{bc}.$$

Therefore,

$$\begin{aligned}\sin\frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}},\\ \cos\frac{A}{2} &= \sqrt{\frac{s(s-a)}{bc}}\\ \tan\frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.\end{aligned} \qquad (12)$$

The student should write the corresponding formulas for $B/2$ and $C/2$.

Let us now inscribe a circle in the given triangle. The center O is equidistant from the three sides; hence it is on the bisectors of the three angles. In Fig. 21 let r be the radius of the inscribed circle and let OB_1, OC_1, and OA_1 be perpendicular, respectively, to CA, AB, and BC.

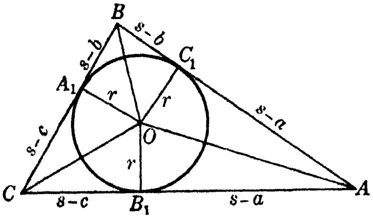

FIG. 21.

We have $CA_1 = CB_1$, $AB_1 = AC_1$, and $BC_1 = BA_1$. *(The two tangents to a circle from an exterior point are equal.)*
Hence,

$$\begin{aligned}2CB_1 + 2AC_1 + 2C_1B &= 2s,\\ CB_1 + AC_1 + C_1B &= s,\\ CB_1 + c \qquad\quad &= s, \ (c = AC_1 + C_1B)\\ CB_1 = s - c.\end{aligned}$$

The length of the tangent from the vertex C to the inscribed circle is $s - c$.

By similar argument (also by cyclic interchange of the letters) we may write

$$\begin{aligned}AC_1 &= s - a,\\ BA_1 &= s - b.\end{aligned}$$

In the right triangle CB_1O, we have

$$\tan\frac{1}{2}C = \frac{r}{s-c},$$

and by cyclic interchange,

$$\begin{aligned}\tan\frac{1}{2}B &= \frac{r}{s-b};\\ \tan\frac{1}{2}A &= \frac{r}{s-a}.\end{aligned}$$

In equations (12) the value of tan $\frac{1}{2}A$ is given as $\tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$. The right member of this equation is equal to $\frac{r}{s-a}$, since each expression gives the value of tan $\frac{1}{2}A$. From this equality, *viz.*,

$$\frac{r}{s-a} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

we obtain a formula for r; it is

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \tag{13}$$

The area of the triangle ABC, Fig. 21, is equal to the sum of the areas of the three triangles AOB, BOC, and COA; *viz.*, $\frac{1}{2}rc + \frac{1}{2}ra + \frac{1}{2}rb = rs$.

$$S = rs. \tag{14}$$

Let the value of r given in (13) be substituted in (14), the result is

$$S = \sqrt{s(s-a)(s-b)(s-c)}, \tag{15}$$

which is the desired formula for the area of a triangle in terms of its three sides. It is a convenient formula for computation, since logarithms may be used with it.

EXERCISE XXIV

1. Compute the areas of the following triangles.

(a) $A = 73°46'$, $b = 295.3$, $c = 209.8$;
(b) $B = 52°21'$, $C = 64°46'$, $a = 146.4$;
(c) $a = 124.6$, $b = 140.2$, $c = 186.5$;
(d) $B = 80°38'$, $a = 267.4$, $c = 568.8$;
(e) $A = 35°42'$, $B = 66°30'$, $c = 21.13$;
(f) $a = 10.82$, $b = 22.66$, $c = 15.44$;
(g) $C = 99°43'$, $a = 24.88$, $b = 61.67$;
(h) $A = 43°43'$, $C = 80°04'$, $b = 8374$;
(i) $a = 4371$, $b = 5540$, $c = 7269$.

2. Compute the radii of the circles inscribed in the triangles in c, f, and i in Problem 1.

3. For the triangle determined by $a = 343.7$, $b = 166.3$, and $c = 455.1$, compute tan $\frac{1}{2}A$, tan $\frac{1}{2}B$, and tan $\frac{1}{2}C$.

4. Prove the formula $S = \frac{1}{2}R^2(\sin 2A + \sin 2B + \sin 2C)$ where R is the radius of the circumscribed circle.

5. Prove the formula $S = abc/4R$.

6. Compute R for the triangle in Problem 3.

SUMMARY

Functions.
  Secant, cosecant, versed sine, coversed sine.
Derivatives.
  Principal value.
Fundamental relations.
Trigonometric identities.
  To prove an identity.
Trigonometric equations.
  To solve an equation.
Area formulas for a triangle.
  In terms of:
    Base and altitude.
    Two sides and the included angle.
    Two angles and the included side.
    Three sides.
    Radius of the inscribed circle and perimeter of the triangle.
Additional formulas for a triangle.

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{r}{s-a}$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

CHAPTER VI

THE ELLIPSE

In previous chapters of the two books in this series studies have been made of the graphs and properties of the following kinds of functions:

- I. Non-algebraic.
  - A. Single valued.
    - a. Trigonometric.
      - sin x, cos x, tan x, cot x, sec x, csc x, vers x, covers x.
    - b. Logarithmic.
      - log x.
    - c. Exponential.
      - a^x.
    - d. Functions whose values are given in tables.
  - B. Multiple valued.
    - The inverses of the trigonometric functions.
- II. Algebraic.
  - A. Single valued.
    - a. Polynomial.
      - $a_0x^n + a_1x^{n-1} + \cdots + a_n$. ($n$ an integer $\geqq 1$).
    - b. Rational.
      - $\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.
  - B. Double valued.
    - Circle.
      - Center at origin; $x^2 + y^2 = r^2$.
      - Center at (h, k); $(x - h)^2 + (y - k)^2 = r^2$.

In this chapter we shall study a double valued algebraic function whose graph is called an **ellipse**. As a preliminary study we need to know how to find on the line determined by two points P_1 and P_2 the point P for which the ratio P_1P/P_1P_2 has a given value.

30. Division of a line segment. *Let P be a point on the line which passes through the two points P_1 and P_2 and let the ratio of the segment P_1P to the given segment P_1P_2 be r. The coordinates of $P(x, y)$ can be expressed in terms of r and the coordinates of P_1 and P_2.* In Fig. 22, it is evident that

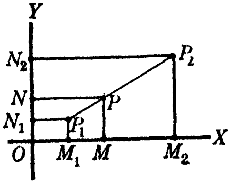

Fig. 22.

$$\frac{M_1M}{M_1M_2} = \frac{P_1P}{P_1P_2} = r,$$
$$\frac{N_1N}{N_1N_2} = \frac{P_1P}{P_1P_2} = r,$$
$$M_1M = rM_1M_2 = r(x_2 - x_1),$$
$$N_1N = rN_1N_2 = r(y_2 - y_1).$$

Hence, since $x = OM = OM_1 + M_1M$ and $y = ON = ON_1 + N_1N$, we have

$$x = x_1 + r(x_2 - x_1),\quad y = y_1 + r(y_2 - y_1). \tag{1}$$

The point P does not need to have a position between P_1 and P_2; it may be on the extension of the line to the right of P_2 or to the left of P_1. In Fig. 23 it is seen that r is negative for one of these positions and positive for the other, since in the one case P_1P and P_1P_2 have the *same sense* on the line, while in the other case P_1P' and P_1P_2 have *opposite senses*. We conclude:

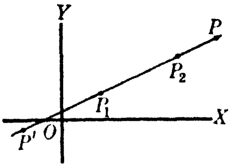

Fig. 23.

1. If r is negative, P lies to the left of P_1, and conversely.
2. If r is positive and less than unity, P lies between P_1 and P_2, and conversely.
3. If r is positive and greater than unity, P lies to the right of P_2, and conversely.

If P_1 and P_2 are interchanged, the above conclusions remain true if the words right and left are also interchanged.

Equations (1) are to be used to determine x and y if P_1, P_2, and r are given. Or either of them may be solved for r, thus determining the ratio of the two segments P_1P and P_1P_2 when the three points are given.

The point P bisects the segment P_1P_2 if $r = \frac{1}{2}$. For this value of r, equations (1) become

$$x = \frac{x_1 + x_2}{2},\quad y = \frac{y_1 + y_2}{2}. \tag{2}$$

EXERCISE XXV

Given $A(0, 0)$, $B(4, 1)$, $C(3, 5)$, $D(1, -2)$:

(*a*) Find the lengths of the line segments BC, BD, AC.

(*b*) Find the mid points of the line segments AB, AD.

(*c*) Find the points on the line segment BC for which $r = 1/3, -1/2, -5/3$, respectively.

(*d*) Find the points on CB for which $r = 1/3, -1/2, -5/3$, respectively.

(*e*) Is the triangle ABC a right triangle?

(*f*) What is the ratio of AP to AD if P is the point $(3, -6)$?

(*g*) Find the equations of the three medians of the triangle ABD. Prove that they meet in a point. Find the point.

(*h*) Prove that the medians of any triangle meet in a point.

(*i*) Write the equation of the line joining the mid points of the sides AB and BD in the triangle ABD. Prove that this line is parallel to AD.

(*j*) There are three positions for the point E so that A, B, D, and E are vertices of a parallelogram; find them.

(*k*) Prove that the diagonals of the parallelogram $ABDE$ bisect each other.

(*l*) The diagonals of any parallelogram bisect each other; prove.

(*m*) The altitudes of a triangle meet in a point; find the point.

(*n*) Write the equation of the line through B which is parallel to CD; put this equation into normal form.

31. The ellipse. If a circular hoop made of fine wire is held in various positions under a light the shadow is sometimes circular (when?) but more often it is oval shaped and looks as if it were made by stretching the circle in opposite directions along a diameter (Fig. 24). This figure, called an **ellipse,** is symmetric with respect to a point O, called the **center,** and with respect to two of its diameters, *viz.*, $A'A$, called the **major diameter** and $B'B$, called the **minor diameter.** All other diameters, as $P'OP$, are longer than $B'B$ and shorter than $A'A$; no one of them is an axis of symmetry.

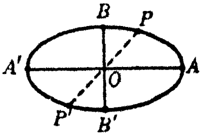

FIG. 24.

The lines obtained by extending $A'A$ and $B'B$ are called, respectively, the **major axis** and the **minor axis** of the ellipse. The points A' and A are called the **vertices** of the curve.

Place the center of the ellipse at the origin of a Cartesian system of coordinates and let OA extend along OX. Denote the length of $A'A$ by $2a$ and of $B'B$ by $2b$. Referred to the coordinate frame described, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \tag{3}$$

i.e., we shall prove that the graph of equation (3) has the properties of the ellipse described above.

Proof. The function y defined by equation (3) is

$$y = \pm\frac{b}{a}\sqrt{a^2 - x^2}.$$

It is a double-valued function of x, symmetric with respect to the x-axis; admissible values of x are $-a \leqq x \leqq a$, for all other values of x the values of y are imaginary. Each of the values

$$y_1 = \frac{b}{a}\sqrt{a^2 - x^2}, \quad y_2 = -\frac{b}{a}\sqrt{a^2 - x^2}$$

is a real continuous function of x for admissible values of x. The x-intercepts are $\pm a$; the function y_1 decreases from b to 0 as x increases from 0 to a.

The student may solve equation (3) for x and state for x conclusions analogous to the ones listed for y. The graph is shown in Fig. 25.

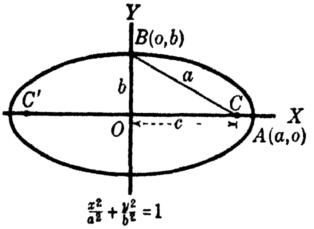

FIG. 25.

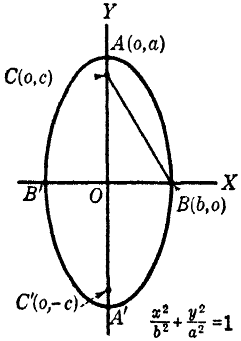

FIG. 26.

Let now the y-axis be the major axis of the ellipse, its equation is then

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \tag{4}$$

and its graph is shown in Fig. 26. In both of the equations (3) and (4) the symbol a represents the length of the semi-major diameter and b the length of the semi-minor diameter. Inspection of the equation of an ellipse whose axes of symmetry are the coordinate axes shows at once whether the major axis is horizontal or vertical.

EXERCISE XXVI

By computing a number of points for each curve, make graphs of the ellipses whose equations are given below. These graphs should be saved for future reference.

(*a*) $\frac{x^2}{25} + \frac{y^2}{9} = 1;$ (*c*) $\frac{x^2}{9} + \frac{y^2}{25} = 1;$

(*b*) $\frac{x^2}{17} + \frac{y^2}{1} = 1;$ (*d*) $\frac{x^2}{36} + \frac{y^2}{16} = 1.$

32. Foci. There are two important points within the ellipse called the **foci**. They lie on the major axis and are found by striking an arc with center B (or B') and radius a (Figs. 25 and 26). The arc must cut $A'A$ twice since a is greater than b. (In a circle where $a = b$ the two foci fall together at the center.)

Let the two foci be denoted by C' and C and let c be the length of OC. In Fig. 25 the coordinates of C' and C are $(-c, 0)$ and $(c, 0)$, respectively; in Fig. 26, they are $(0, -c)$ and $(0, c)$, respectively. In both cases

$$c^2 = a^2 - b^2. \tag{5}$$

EXERCISE XXVII

1. In the graphs in Exercise XXV locate the foci and compute their coordinates

2. Write the equations and construct the graphs of the ellipses for which

(*a*) $a = 6, b = 2$; (*b*) $a = 6, b = 4$; (*c*) $a = 6, b = 5$;
(*d*) $a = 6, b = 5\frac{1}{2}$; (*e*) $a = 6, b = \frac{1}{2}$; (*f*) $a = 4, c = 3$;
(*g*) $a = 8, c = 3$; (*h*) $a = 8, c = 5$; (*i*) $a = 8, c = 7$.

Let the major axis extend along the x-axis and locate the foci

33. The string property of the ellipse. Let the ends of a string whose length is $2a$ units be fastened (say with thumb tacks) at two points C' and C whose distance apart is $2c$ units. With a pencil pulling the string taut an ellipse can be traced as in Fig. 25. To prove that this is an ellipse we must show that with a suitable choice of a frame of reference the coordinates of the general point $P(x, y)$ on the curve satisfy equation (3).

Proof. Let the mid point of the line segment $C'C$ be the origin of coordinates and let OX extend along OC. From the construction, we have, for every point P on the curve,

$$C'P + CP = 2a.$$

The analytical expression for the length of $C'P$ is $\sqrt{(x + c)^2 + y^2}$ and of CP is $\sqrt{(x - c)^2 + y^2}$, since the coordinates of C' and C are $(-c, 0)$ and $(c, 0)$, respectively. When these values are substituted in the equation $C'P = 2a - CP$, the result is

$$\sqrt{(x + c)^2 + y^2} = 2a - \sqrt{(x - c)^2 + y^2},$$

from which (the following steps are self-explanatory):

$$(x + c)^2 + y^2 = 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + (x - c)^2 + y^2,$$

$$a^2 - cx = a\sqrt{(x - c)^2 + y^2}, \tag{6}$$

$$(a^2 - cx)^2 = a^2(x - c)^2 + a^2y^2,$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2),$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

The last equation is identical with equation (3), since from (5) we have $b^2 = a^2 - c^2$. *q.e.d.*

EXERCISE XXVIII

1. Construct with a string the ellipses in Exercise XXVI, using the same units as were used before. Compare the two sets of curves.

2. Construct with a string the ellipses in Exercise XXVII, (*a*) to (*e*) inclusive. For each of these ellipses the length of the string is 12 but the shape of the ellipse changes when the distance between the points C' and C changes.

34. Eccentricity. The shape of an ellipse depends on the relative lengths of a and b or of a and c; for example, the two ellipses for which a and c have the values $a = 6$, $c = 2$ and $a = 15$, $c = 5$ have the same shape as can be seen if they are constructed with a string. In both cases the ratio of c to a is $\frac{1}{3}$. The **eccentricity** of an ellipse, denoted by the letter e, is defined by the equation

$$e = \frac{c}{a}, \tag{7}$$

and *two ellipses with the same eccentricity have the same shape.*

Since, from equation (7), $e = \sqrt{a^2 - b^2}/a$, we see that for $b \neq 0$ the eccentricity e of an ellipse is always less than unity. It is zero when and only when $a = b$, *i.e.*, when the ellipse is a circle.

From equation (6) on page 73 can be obtained a simple formula for the length of $C'P$ and CP (called the **focal radii** of the point $P(x, y)$). Let c in the left member be replaced by ae (equation 7); then since the radical on the right gives the length of CP we get, after dividing both members by a,

$$CP = a - ex.$$

Also, since $C'P = 2a - CP$, we have

$$C'P = a + ex,$$

i.e., the lengths of the focal radii *to the point whose abscissa is x are $a - ex$ and $a + ex$.*

35. Equation of the tangent to an ellipse at a given point. The equation of the line which is tangent to an ellipse at the point $P_1(x_1, y_1)$ is given by

$$y - y_1 = m(x - x_1),$$

where m is the value of the y-derivative at the point. To compute the derivative we differentiate both members of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with respect to x and solve for $D_x y$. This gives

$$\frac{2x}{a^2} + \frac{2y}{b^2}D_x y = 0,$$

$$D_x y = -\frac{b^2 x}{a^2 y}. \tag{8}$$

The equation of the tangent line at P_1, *viz.*,

$$y - y_1 = -\frac{b^2 x_1}{a^2 y_1}(x - x_1),$$

can be cleared of fractions and simplified in form by replacing $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$ by 1. This is permissible since P_1 is on the ellipse and therefore its coordinates must satisfy the equation of the ellipse. The result of the simplification gives

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1, \tag{9}$$

the equation of the line through P_1 *which is tangent to the ellipse whose equation is* (3).

EXERCISE XXIX

1. Write the equation of the line through P_1 which is tangent to the circle $x^2 + y^2 = r^2$.

2. Write the equations of the tangents to $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at the following points $P_1\left(3, \frac{12}{5}\right)$, $P_2\left(3, -\frac{12}{5}\right)$, $P_3\left(-3, \frac{12}{5}\right)$, $P_4(0, 3)$; $P_5\left(-2, \frac{3}{5}\sqrt{21}\right)$. Construct the ellipse and make the graphs of the lines whose equations you have computed.

3. What is the eccentricity of the ellipse in the last problem?

4. Compute the lengths of the focal radii to P_1, P_2, and P_4 in Problem 4.

36. Tangents to an ellipse with a given slope. It is evident that when a slope is given there will be two of the tangents to a given ellipse having this slope. Furthermore, it will be proved that *parallel tangents touch the ellipse at points which are ends of a diameter.*

Let the given slope be denoted by m and let equation (3) be the equation of the ellipse. We wish to find two lines $y - y_1 =$

$m(x - x_1)$, *i.e.*, $y - mx = y_1 - mx_1$, which are tangent to (3). We must therefore find two points (x_1, y_1) for which we have simultaneously

$$-\frac{b^2x_1}{a^2y_1} = m, \text{ and } \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

When the value of x_1 obtained from the first equation is substituted in the second equation, the values of y_1 are found to be $y_1 = \pm\frac{b^2}{\sqrt{a^2m^2+b^2}}$; whence $x_1 = \mp\frac{a^2m}{\sqrt{a^2m^2+b^2}}$. Thus the two points at which the tangents with slope m touch the ellipse are

$$P_1\left(\frac{-a^2m}{\sqrt{a^2m^2+b^2}}, \frac{b^2}{\sqrt{a^2m^2+b^2}}\right) \text{and } P_2\left(\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \frac{-b^2}{\sqrt{a^2m^2+b^2}}\right).$$

The equation of the tangent at the first of these points is found to be $y - mx = \sqrt{a^2m^2+b^2}$, while $y - mx = -\sqrt{a^2m^2+b^2}$ is the equation of the tangent at P_2.

The two points of tangency are symmetric with respect to the origin* and are therefore the ends of a diameter, as was asserted.

The slope of the diameter P_1P_2 is seen to be $\frac{-b^2}{a^2m}$; its equation is $y = \frac{-b^2}{a^2m}x$, or,

$$\mathbf{b^2x + a^2my = 0.} \tag{10}$$

In Fig. 27 the lines L_1 and L_2 are parallel; their slope is m; hence the slope of P_2P_1 is $-b^2/a^2m$. The equation

$$y = mx + \sqrt{a^2m^2+b^2}$$

is the equation of L_1; the equation $y = mx + k$ where m is the (given) slope of L_1 and where k is a parameter is the equation of the pencil of lines parallel to L_1. For suitable values of k,† these lines cut the ellipse in two real distinct points—points which are the ends of chords parallel to L_1 and L_2. We shall prove that *each chord of the system of chords with slope m is bisected by the*

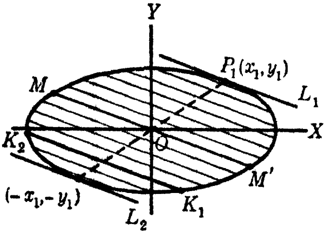

FIG. 27.

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* *Cf.* Vol. I, p. 188.

† The values of k for which $y = mx + k$ cut the ellipse in two distinct points are given by $-\sqrt{a^2m^2+b^2} < k < \sqrt{a^2m^2+b^2}$. Why? These two points are real for every m.

diameter P_2P_1 whose slope is $-b^2/a^2m$; whence, because of symmetry, each chord whose slope is $-b^2/a^2m$ is bisected by the diameter whose slope is m.

Proof. The abscissas of the two points of intersection of the line $y = mx + k$ with the ellipse (3) are obtained by solving for x the equation

$$\frac{x^2}{a^2} + \frac{(mx + k)^2}{b^2} = 1.$$

The ordinates of the two points are obtained by substituting in the equation $y = mx + b$ the abscissas determined in the last equation. The two points are found to be

$$\left(\frac{-a^2mk \pm ab\sqrt{b^2 + a^2m^2 - k^2}}{a^2m^2 + b^2}, \frac{b^2k \pm amb\sqrt{b^2 + a^2m^2 - k^2}}{a^2m^2 + b^2}\right). \qquad (11)$$

The mid point of the chord whose end points are given by (11) is found to be $\left(\frac{-a^2mk}{a^2m^2 + b^2}, \frac{b^2k}{a^2m^2 + b^2}\right)$ and it is easily seen that this point lies on the diameter P_2P_1 whose equation is given by (10). *q.e.d.*

The two diameters MM' and P_2P_1 with respective slopes m and $-b^2/a^2m$ and which have the property just proved, *viz.*, that each one bisects all of the chords which are parallel to the other one, are called **conjugate diameters**; the systems of chords parallel to conjugate diameters are called **conjugate systems of chords.**

Theorem. *Two systems of chords are conjugate if their respective slopes are m and $-b^2/a^2m$.*

EXERCISE XXX

1. Find the equations of the two lines with slope $\frac{1}{2}$ which are tangent to $\frac{x^2}{36} + \frac{y^2}{16} = 1$. *Ans.* $y = \frac{1}{2}x + 5$ and $y = \frac{1}{2}x - 5$.

2. Find the points at which the tangents in Problem 1 meet the ellipse. *Ans.* $(-\frac{18}{5}, \frac{16}{5})$ and $(\frac{18}{5}, -\frac{16}{5})$.

3. Write the equation of the system of chords with slope $\frac{1}{2}$. (By equation of the system of chords is meant the equation of the lines obtained by extending the chords indefinitely.)

4. Write the equation of the diameter of the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$ which bisects the chords in Problem 3.

5. Write the equation of the diameter which is conjugate to the diameter in Problem 4.

6. Do formulas (10) and (11) hold for the ellipse whose equation is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1?$$

7. Given the ellipse whose equation is $\frac{x^2}{9} + \frac{y}{25} = 1$. Find the diameter which bisects the system of chords whose slope is -2; find the points of tangency of the lines with slope -2 which touch the given ellipse. Write the equations of the tangent lines.

37. The reflection property of the ellipse. When a sound wave or a light wave strikes a surface, the angle between the direction of the wave and the **normal* to the surface** is called *the angle of incidence* of the wave. After meeting the surface, the wave is reflected in a direction which makes an angle with the normal called the *angle of reflection.* In physics it is proved that for a given wave these two angles are equal. In the figure the angles of incidence and reflection are designated by i and r, respectively. The normal at P, the point of incidence, is PR.

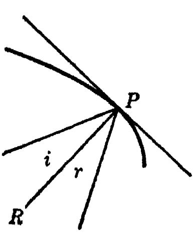

FIG. 28.

The reflection property of the ellipse is expressed by the theorem: *the two focal radii to P make equal angles with the normal at P.*

This means that when a source of light or sound is placed at one focus C', the waves meeting the ellipse at various points P are *all* reflected to C.

Proof. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, let the inclination of the tangent at $P'(x', y')$ be denoted by τ, and let the incident ray from C' to P' make an angle i with the normal NP' (Fig. 29). Join C to P' and denote the angle $NP'C$ by α. If we can prove angle α = angle i, we shall have proved that the ray $C'P'$ is reflected in the direction $P'C$.

Since each of the angles i and α is an acute angle, it is sufficient to prove $\sin \alpha = \sin i$.

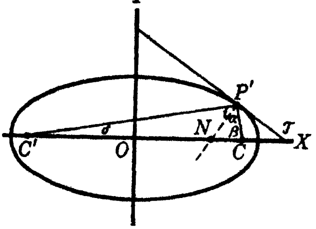

FIG. 29.

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* The **normal to a curve** at a point P is the line through P which is perpendicular to the tangent line at P. All of the lines which are tangent to a surface at P lie in a plane called a **tangent plane.** The **normal to a surface** at a point P is the line through P which is perpendicular to the tangent plane at P.

From the law of sines we have

$$\sin i = \frac{C'N}{NP'}, \sin \delta = \frac{C'N}{NP'} \cdot \frac{y'}{C'P'} = \frac{C'N}{NP'} \cdot \frac{y'}{a + ex'},$$
$$\sin \alpha = \frac{NC}{NP'}, \sin \beta = \frac{NC}{NP'} \cdot \frac{y'}{CP'} = \frac{NC}{NP'} \cdot \frac{y'}{a - ex'}.$$

Therefore

$$\frac{\sin i}{\sin \alpha} = \frac{C'N}{NC} \cdot \frac{a - ex'}{a + ex'} = \frac{c + ON}{c - ON} \cdot \frac{a - ex'}{a + ex'}. \tag{12}$$

The length ON can be computed by finding the abscissa of N, *i.e.*, the x-intercept of the normal NP'. The equation of the normal is

$$y - y' = \frac{a^2 y'}{b^2 x'}(x - x');$$

the x-intercept is found by setting $y = 0$ and solving for x. The result is $x = \frac{c^2}{a^2}x' = e^2x' = ON$. When this value is put in (12) we get (after replacing c by ae)

$$\frac{\sin i}{\sin \alpha} = \frac{ae + e^2x'}{ae - e^2x'} \cdot \frac{a - ex'}{a + ex'} = 1.$$

Hence $\sin i = \sin \alpha$ and therefore the ray $C'P'$ is reflected in the direction $P'C$. *q.e.d.*

The reflection property of the ellipse furnishes *a simple means of constructing with ruler and compasses the tangent to a given ellipse at an arbitrary point P*. We draw the two focal radii $C'P$ and CP and bisect the angle $C'PC$, thus obtaining the normal to the curve at P. It only remains to construct the line which is perpendicular to the normal NP at P.

The reflection property of the ellipse is utilized in the construction of "whispering galleries." The gallery has the form of an ellipsoid. It has the property that any plane through two fixed points C' and C cuts the ellipsoid in an ellipse which has C' and C as foci. If a person at one focus drops a pin or whispers, the sound waves are all reflected to the other focus and an auditor at that point gets a surprising volume of sound.

EXERCISE XXXI

1. Construct geometrically the tangent to $\frac{x^2}{17} + \frac{y^2}{1} = 1$ at the point P' whose abscissa is 4 and whose ordinate is negative.

2. Write the equation of the tangent in Problem 1.

3. Write the equation of the normal at the same point.

4. Write the equations of the two focal radii to this point.

5. Compute the lengths of the two focal radii. What is the sum of the lengths?

6. Write the equation of the diameter through P'. Of its conjugate diameter.

7. Construct the ellipse whose equation is $\frac{x^2}{2} + \frac{y^2}{18} = 1$ and prove the following statements:

(*a*) The equation of the tangent at $(1, -3)$ is $3x - y - 6 = 0$.

(*b*) The equation of the chords parallel to this tangent is $3x - y + k = 0$.

(*c*) The diameter which is conjugate to the system of chords in (*b*) is $3x + y = 0$.

(*d*) The equation of the normal at $(1, -3)$ is $x + 3y + 8 = 0$.

8. For the ellipse in the last problem compute $\sin \iota$ where ι is the angle between $C' P$ and the normal at $P(1, -3)$.

9. Write the equations of the focal radii to $P(1, -3)$.

SUMMARY

- Ellipse.
  - Axes.
    - Major.
    - Minor.
  - Center, vertices.
  - Foci, focal radii, eccentricity.
  - Type form of equation.
  - Equation of the tangent at a given point.
  - Equation of the tangents with a given slope.
  - Conjugate diameters, parallel chords.
  - The string property of the ellipse.
  - The reflection property of the ellipse.
    - Construction of the tangent at a given point.
- Division of a line segment.

CHAPTER VII

THE HYPERBOLA

With a suitable choice of a coordinate frame of reference the equation of the hyperbola differs from the equation of the ellipse only in the sign of one term. The two curves have many properties in common, though in other respects they differ widely.

38. The hyperbola. The graph of the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{1}$$

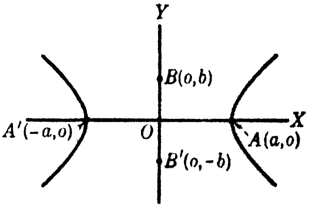
is called the **hyperbola.** For every admissible x there are two values of y which are equal in numerical value but are opposite in sign. They are

$$y = \pm b\sqrt{\frac{x^2}{a^2} - 1}. \tag{2}$$

Hence, *the x-axis is an axis of symmetry.* No real part of the graph lies between the vertical lines $x = -a$ and $x = a$ since for $-a < x < a$ the radicand in equation (2) is negative.

For every y there are two values of x which are numerically equal but opposite in sign. They are

$$x = \pm a\sqrt{1 + \frac{y^2}{b^2}}.$$



$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

FIG. 30.

Hence, *the y-axis is an axis of symmetry.* The intersection of the two axes of symmetry is a point of symmetry. This point is called the **center** of the hyperbola. Equation (2) shows that the positive value of y, *viz.*, $y = b\sqrt{\frac{x^2}{a^2} - 1}$, increases as x increases and is continuous. The graph is shown in Fig. 30. The points $A(a, 0)$ and $A'(-a, 0)$ are **vertices** of the hyperbola. They are the intersections of the curve with its **transverse axis** which in Fig. 30 is the x-axis. The y-axis here is the second axis of symmetry; it is called the **conjugate axis.** The length of the line segment $A'A$ is $2a$. The points $B(0, b)$ and $B'(0, -b)$, though they are not on the curve, play an important part in developing *the properties of the hyperbola.*

Construct horizontal lines through B and B' and vertical lines through A and A'. Let two of the intersections be M and M', as indicated in Fig. 31, where $OM = \sqrt{a^2+b^2} = OM'$, the slope of OM is $\frac{b}{a}$ and its equation is $y = \frac{b}{a}x$; the slope of OM' is $\frac{-b}{a}$ and its equation is $y = \frac{-b}{a}x$. With O as center and OM as radius, strike an arc cutting the transverse axis in C' and C. These points are the two **foci of the hyperbola** and have properties analogous to the ones proved for the ellipse. They will be discussed in later sections. The coordinates of C' and C will be denoted by $(-c, 0)$ and $(c, 0)$ where c is defined by the equation

$$c^2 = a^2 + b^2. \tag{3}$$

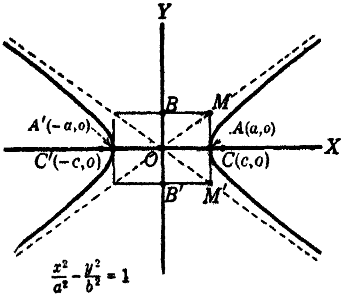


FIG. 31.

39. The string property of the hyperbola. Fasten thumb tacks at C' and C (Fig. 32); tie a thread to a pencil point, pass the thread around C' and C as indicated in the figure, hold the free ends of the thread firmly together at H, keep the thread taut at P, and allow H to move. As P moves on one branch we have always $C'P - CP =$ a constant value, while, for the other branch, $CP - C'P =$ the same constant value. Let us denote this value by $2a$ and it can be shown that the coordinates of P satisfy equation (1).

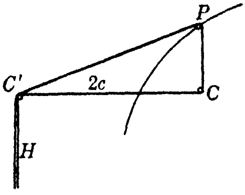


FIG 32.

Proof. For the two branches we have $C'P = CP + 2a$ and $C'P = CP - 2a$. Let the mid point of the line $C'C$ be the origin of coordinates and let OX extend along OC. The lengths $C'P$ and CP are, respectively, $\sqrt{(x+c)^2+y^2}$ and $\sqrt{(x-c)^2+y^2}$. For the two branches we have

$$\sqrt{(x+c)^2+y^2} = \sqrt{(x-c)^2+y^2} \pm 2a.$$

When this equation is squared, like terms collected, rational terms transposed to the left side of the equality sign, and both members divided by 4, the result is

$$cx - a^2 = \pm a\sqrt{(x-c)^2+y^2}; \tag{4}$$

whence

$$c^2x^2 - 2a^2cx + a^4 = a^2(x - c)^2 + a^2y^2,$$
$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2).$$

Let the symbol b^2 replace $c^2 - a^2$. Division by a^2b^2 now gives

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad \textit{q.e.d.}$$

By increasing or decreasing the length $2c$ (or $2b$), leaving $2a$ fixed, or by increasing or decreasing $2a$, leaving $2c$ (or $2b$) fixed, the student may construct a number of hyperbolas, noting the effect of a change in the constants on the shape of the hyperbola. For example, the hyperbola widens if c increases while a remains fixed, and it narrows as c decreases. As in the case of the ellipse the shape of the hyperbola depends on c/a which again is called the **eccentricity** and is denoted by e.

$$\frac{c}{a} = e.$$

Since c is always longer than a, *the eccentricity of the hyperbola is greater than unity.*

Equation (4) with c replaced by ae becomes

$$ex - a = \pm CP,$$

and since $C'P - CP = \pm 2a$, we have

$$ex + a = \pm C'P.$$

The student can easily satisfy himself that the upper sign is correct in the last two equations if P is on the right-hand branch of the hyperbola and the lower sign is correct if P is on the left-hand branch. (*Cf.* the proof of the string property.) In all cases we can make the following statement; *the numerical values of the lengths of the focal radii from C and C′ respectively, to the point P are given by* $|ex - a|$ *and* $|ex + a|$.

40. Equation of the tangent to a given hyperbola at a given point *P*. By paralleling the computation in Sec. 35, the following results are obtained for the hyperbola whose equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

1. The derivative is $D_xy = \frac{b^2x}{a^2y}$,
2. The equation of the tangent at $P_1(x_1, y_1)$ is

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1. \qquad (5)$$

3. The two tangents whose slope is m are given by the equations $y - mx = \pm\sqrt{a^2m^2 - b^2}$. (See page 85 in regard to the values of m which correspond to *real* tangents.)

4. The points at which lines with slope m are tangent to the hyperbola are

$$\left(\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{b^2}{\sqrt{a^2m^2 - b^2}}\right) \text{ and } \left(\frac{-a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{-b^2}{\sqrt{a^2m^2 - b^2}}\right).$$

EXERCISE XXXII

1. Write the equations of the tangents to $\frac{x^2}{36} - \frac{y^2}{25} = 1$ at the two points whose abscissa is (-8)

2. What is the eccentricity of the hyperbola in Problem 1?

3. Compute the lengths of the focal radii to the points in Problem 1

4. After plotting the hyperbola $\frac{x^2}{25} - \frac{y^2}{36} = 1$, discuss the effect of interchanging a^2 and b^2. Is the effect similar to that in the case of the ellipse?

41. Conjugate hyperbolas. The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \tag{6}$$

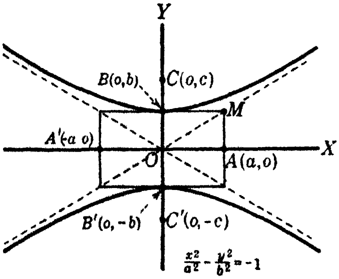


FIG 33.

differs from equation (1) only in the sign of the constant term. The graph of (6) is also a hyperbola; its x-intercepts are imaginary and its y-intercepts are real; *i.e.*, its transverse axis is vertical and its conjugate axis is horizontal. The graph is shown in Fig. 33. The vertices of the hyperbola are $B(0,b)$ and $B'(0, -b)$ and the foci C and C' lie on the y-axis. They are constructed as before by drawing a circle whose center is O and radius $OM = \sqrt{a^2 + b^2}$. The two hyperbolas whose equations are (1) and (6) are called **conjugate hyperbolas**; *the transverse axis of the one is the conjugate axis of the other.* We shall continue to use the symbol a for x-intercept in the one case and b for y-intercept in the other (contrary to our usage in the case of the ellipse). With this understanding it is clear that the eccentricity of the hyperbola in Fig. 33 is c/b and that certain formulas which were established for the ellipse whose transverse axis is horizontal must be appropriately modified before they can be applied to the conjugate hyperbola. The results are given below; proofs which are not given should be supplied by the student.

1. The tangent to the hyperbola whose equation is (6) at the point P_1 (x_1, y_1) is $\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = -1$.

2. The two lines whose slope is m which are tangent to (6) are the lines whose equations are

$$y - mx = \pm\sqrt{b^2 - a^2m^2}; \tag{7}$$

whereas it was shown on page 84 that the tangents with slope m to the hyperbola whose equation is (1) are

$$y - mx = \pm\sqrt{a^2m^2 - b^2}. \tag{8}$$

An examination of the radicands in the last two equations shows that the lines in equation (8) are real only when m^2 is greater than b^2/a^2 while the lines in equation (7) are real only when m^2 is less than b^2/a^2. Thus not every number can be the slope of a line which is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and, likewise, not every number can be the slope of a line which is tangent to its conjugate hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

The lines whose equations are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$, *i.e.*, the lines for which $m^2 = b^2/a^2$, have a very special relation to the two hyperbolas. They are discussed in the next section.

42. The asymptotes of the hyperbola. The lines whose equations are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ intersect at the origin of coordinates dividing the plane into four parts; two of these parts contain the two branches of the hyperbola whose equation is (1); the remaining two parts contain the two branches of the conjugate hyperbola. *There is no finite intersection of either line with these curves.*

Proof. Because of the symmetry of the figures involved, it is only necessary to prove the above statement for the part of the curve shown in the accompanying figure. Let P with coordinates (x, y) be any point on the hyperbola, and let Q be the point on the line $y = \frac{b}{a}x$ whose abscissa is the same as the abscissa of P. The

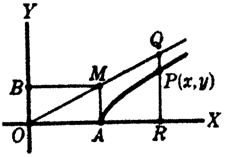


FIG. 34.

ordinate of P (RP in Fig. 34) is equal to $b\sqrt{\frac{x^2}{a^2} - 1}$ and the ordinate of Q is equal to $\frac{b}{a}x$. It is clear that $b \cdot \frac{x}{a}$ is greater than $b\sqrt{\frac{x^2}{a^2} - 1}$ for all positive values of x greater than a, since $\sqrt{\frac{x^2}{a^2} - 1}$ is less than $\frac{x}{a}$. Hence the hyperbola can never cross the line. *q.e.d.*

Moreover, the greater the value of x, the smaller becomes the difference (PQ in Fig. 34) between the two ordinates and this difference, *viz.*, $b\left[\frac{x}{a} - \sqrt{\frac{x^2}{a^2} - 1}\right]$, can be made smaller than any previously assigned small positive number by taking x sufficiently large.

Proof.

$$\frac{b}{a}[x - \sqrt{x^2 - a^2}] = \frac{b}{a}[x - \sqrt{x^2 - a^2}]\frac{x + \sqrt{x^2 - a^2}}{x + \sqrt{x^2 - a^2}} = \frac{ab}{x + \sqrt{x^2 - a^2}},$$

and the truth of the statement is evident.

The line $y = \frac{b}{a}x$ is called an **asymptote** of the hyperbola and the curve is said to approach the line asymptotically. The same line is an asymptote of the conjugate hyperbola and the line $y = -\frac{b}{a}x$ is also an asymptote of each of the hyperbolas.

Consider again equation (8), $y - mx = \pm\sqrt{a^2m^2 - b^2}$, the equation of the two lines with slope m which are tangent to the hyperbola. The y-intercepts of these two tangents are $\pm\sqrt{a^2m^2 - b^2}$. As $m \to +\frac{b}{a}$, the y-intercepts approach zero and the two tangents have as a limiting position the asymptote $y - \frac{b}{a}x = 0$. Similarly, as $m \to -\frac{b}{a}$, the y-intercepts approach zero and the two tangents have as a limiting position the asymptote $y + \frac{b}{a}x = 0$. This shows analytically the conclusion which follows from the proof that the curve approaches the two lines asymptotically; *viz.*, an asymptote of the hyperbola is the limiting position of a tangent line as the point of tangency moves to infinity. More briefly, we say *the asymptotes are tangent to the hyperbola at infinity.*

In sketching the graph of a hyperbola, only a few points need be plotted if the asymptotes are first drawn carefully. The single equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, which is obtained from equation (1) and from equation (6) by replacing the constant in the right member by zero, is the equation of the two asymptotes of the two hyperbolas. For the left member of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ factors into two linear factors $\frac{x}{a} - \frac{y}{b}$ and $\frac{x}{a} + \frac{y}{b}$ so that the given equation may be written

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 0.$$

This equation is satisfied by any pair of numbers (x, y) which makes the first factor zero and is satisfied by any pair of numbers (x, y) which makes the second factor zero; *i.e.*, it is satisfied by the points on the two asymptotes

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ and } \frac{x}{a} + \frac{y}{b} = 0.$$

No point not on one of the asymptotes can make $\frac{x^2}{a^2} - \frac{y^2}{b^2}$ equal to zero, since a product cannot be zero unless one of the factors is zero.

EXERCISE XXXIII

1. Draw the graphs of the following hyperbolas and write the equations of their asymptotes:

(*a*) $\frac{x^2}{16} - \frac{y^2}{25} = 1;$ (*b*) $\frac{x^2}{16} - \frac{y^2}{9} = 1;$

(*c*) $\frac{x^2}{49} - \frac{y^2}{4} = -1;$ (*d*) $\frac{x^2}{100} - \frac{y^2}{64} = -1.$

2. Draw the graphs of the hyperbolas which are the conjugates of the hyperbolas in Problem 1.

3. Which of the hyperbolas in Problems 1 and 2 have tangents whose slope is ½? ⁵⁄₂? 8? Write the equations of these tangents.

4. Write the equation of the tangent at $(-4\sqrt{5}, 10)$ to the hyperbola whose equation is (*a*) in Problem 1.

5. Is the line $4x + 15y - 80 = 0$ tangent to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$? If so, where?

6. What is the eccentricity of the hyperbola whose equation is $\frac{x^2}{25} - \frac{y^2}{16} = -1$? Determine k so that $y = x + k$ is tangent to the hyperbola.

Ans. $k = \pm\sqrt{11}$.

43. Parallel chords. Conjugate diameters. In Fig. 35 the lines L_1 and L_2 have slope $m > b/a$ and are drawn tangent to the hyperbola whose equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Of the lines with this slope, *viz.*, $y = mx + k$, some cut the right-hand branch of the hyperbola in two distinct points, some cut the left-hand branch in two distinct points, while some have no real intersections with the curve. The segments of those lines actually intersecting the hyperbola, as $M'M$, $S'S$, . . . , are called **chords** of the hyperbola.

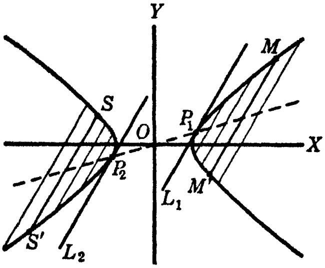


FIG. 35.

The equation of the diameter P_2OP_1 is

$$b^2x - a_2my = 0 \tag{9}$$

and its slope is b^2/a^2m. (*Cf.* equation (10) on page 76.) It can be shown that if the diameter P_2P_1 is extended it meets each of the chords whose slope is m at its midpoint. From the symmetry of the figure the chords whose slope is b^2/a^2m, *i.e.*, the chords of the *conjugate* hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, which are parallel to P_2P_1 are bisected by the diameter of the conjugate hyperbola whose slope is m. Thus *m and b^2/a^2m are slopes of conjugate systems of parallel chords* and the diameters with these slopes are called **conjugate diameters.**

EXERCISE XXXIV

1. For the hyperbola $\frac{x^2}{16} - \frac{y^2}{25} = 1$, find the equations of the two tangents with slope $m = -{}^{13}\!/_{4}$.

2. Find the equation of the diameter of the hyperbola in Problem 1 which bisects the system of chords with slope $m = -{}^{13}\!/_{4}$.

3. Find the equation of the diameter which is conjugate to the diameter in the last problem.

4. Find the equation of the tangent to $\frac{x^2}{25} - \frac{y^2}{16} = -1$ at $\left(-12, \frac{52}{5}\right)$.

5. Find the equation of the tangent to the lower branch of $\frac{x^2}{25} - \frac{y^2}{16} = -1$ which is parallel to the tangent in the last problem.

6. Write the equations of two conjugate diameters of $\frac{x^2}{25} - \frac{y^2}{16} = +1$ and $\frac{x^2}{25} - \frac{y^2}{16} = -1$ if the slope of one of the diameters is $-\frac{1}{3}$.

44. The reflection property of the hyperbola. The reflection property of the hyperbola is expressed by the theorem: *the two focal radii drawn to any point P on a hyperbola make equal angles with the tangent at P.*

Proof In Fig 36, $P'(x', y')$ is a point on the hyperbola whose equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the inclinations of the focal radii CP' and $C'P'$ are denoted by γ and δ, respectively; the tangent $P'L$ at P' makes angles α and β with the focal radii as indicated in the drawing, and the inclination of $P'L$ is τ We wish to prove angle α equal to angle β

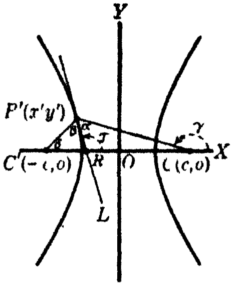


FIG 36.

Since each of the angles α and β is an acute angle, it is sufficient to prove $\sin \alpha = \sin \beta$

From the law of sines we have

$$\sin \alpha = \frac{RC}{RP'} \sin \gamma = \frac{RC}{RP'} \frac{y'}{CP'} = \frac{RC}{RP'} \frac{y'}{a - ex'};$$

$$\sin \beta = \frac{C'R}{RP'} \sin \delta = \frac{C'R}{RP'} \frac{y'}{C'P'} = \frac{C'R}{RP'} \frac{y'}{-(ex' + a)}$$

Therefore,

$$\frac{\sin \alpha}{\sin \beta} = \frac{RC}{C'R} \frac{-(ex' + a)}{a - ex'} = \frac{c + RO}{c - RO} \frac{ex' + a}{ex' - a} \tag{10}$$

The length RO can be computed by finding the abscissa of R, the x-intercept of PL The equation of PL is $\frac{x'x}{a^2} - \frac{y'y}{b^2} = 1$, its x-intercept is $x = \frac{a^2}{x'}$ Hence $RO = -OR = -\frac{a^2}{x'}$, and $c + RO = ae - \frac{a^2}{x'} = \frac{a(ex' - a)}{x'}$ while $c - RO = ae + \frac{a^2}{x'} = \frac{a(ex' + a)}{x'}$ When these values are substituted in equation (10), the result is $\frac{\sin \alpha}{\sin \beta} = 1$ *q e d*

EXERCISE XXXV

1. Describe a method of constructing with ruler and compasses the tangent to a hyperbola at a given point

2. Construct the tangents to the hyperbola $\frac{x^2}{15} - \frac{y^2}{6} = 1$ at the points (5, 2) and (5, −2) Find the lengths of the focal radii to the point (5, −2)

3. Devise a method of constructing the diameter of a hyperbola which is conjugate to a given diameter

4. Use the method of Problem 3 to construct the diameter of $\frac{x^2}{64} - \frac{y^2}{100} = 1$ which is conjugate to the diameter of $\frac{x^2}{64} - \frac{y^2}{100} = -1$ and whose slope is $\frac{2}{3}$ Write the equations of all of the lines used in the construction.

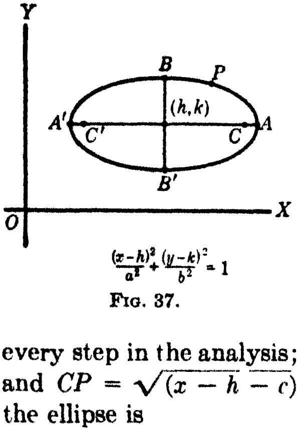
45. The rectangular hyperbola. When a and b are equal, the equations of the two conjugate hyperbolas are $x^2 - y^2 = a^2$ and $x^2 - y^2 = -a^2$. For the hyperbola whose equation is $x^2 - y^2 = a^2$ we note the following:

1. The x-intercepts are $\pm a$.
2. The asymptotes, $y = \pm x$, are mutually perpendicular.
3. The foci are $C(a\sqrt{2}, 0)$ and $C'(-a\sqrt{2}, 0)$.
4. The eccentricity is equal to $\sqrt{2}$.
5. The equation of the tangent at (x_1, y_1) is

$$x_1x - y_1y = a^2.$$

The given hyperbola is congruent to its conjugate. These hyperbolas are called **rectangular hyperbolas** and are also called **equilateral hyperbolas.**

46. Ellipses and hyperbolas with centers not at the origin.



$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

FIG. 37.

Let the center of an ellipse be at the point (h, k) and let its axes be parallel to the x- and y-axes as in Fig. 37. The equations of the two axes of the ellipse are $x = h$ and $y = k$. The foci are $C'(h - c, k)$ and $C(h + c, k)$. By the string property of the ellipse we have $C'P + CP = 2a$ for every point P on the ellipse. Now the analytical expression for $C'P + CP = 2a$ is obtained from the one in Sec. 33 by replacing x by $x - h$ and y by $y - k$ at every step in the analysis; *viz.*, $C'P = \sqrt{(x - h + c)^2 + (y - k)^2}$ and $CP = \sqrt{(x - h - c)^2 + (y - k)^2}$. Hence the equation of the ellipse is

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1, \tag{11}$$

and the tangent at $P_1(x_1, y_1)$ is

$$\frac{(x_1 - h)(x - h)}{a^2} + \frac{(y_1 - k)(y - k)}{b^2} = 1.$$

Similarly, if the hyperbola has its center at (h, k) and its axes are parallel to the axes of coordinates, its equation is

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1, \tag{12}$$

or

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1. \tag{13}$$

47. The latus rectum. A chord of an ellipse which passes through a focus and is perpendicular to the major axis is called a **latus rectum.** The length of a latus rectum is easily found by solving simultaneously with the ellipse the line through a focus which is perpendicular to the major axis. If a is greater than b in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the line under discussion is given by the equation $x = c$. It cuts the ellipse at two points whose ordinates are b^2/a and $-b^2/a$, respectively. Hence *the length of this latus rectum is* $2b^2/a$. Obviously, the latus rectum through C' is equal in length to the one through C.

The same result is obtained by solving the equation $y = c$ simultaneously with $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 (a > b)$.

A latus rectum of a hyperbola is the chord through a focus which is perpendicular to the transverse axis.

EXERCISE XXXVI

1. Compute the length of a latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and of its conjugate hyperbola.

2. Compute the length of a latus rectum of an equilateral hyperbola.

3. Write the equations of the following ellipses:

(*a*) Center at (2, 1), major diameter = 5, minor diameter = 2; the major axis is horizontal.

(*b*) Center at (−2, 3), major diameter = 3, minor diameter = 1; the major axis is vertical.

(*c*) Center at (−1, −4), major diameter = 4, minor diameter = 2 There are two solutions.

4. Locate the foci and compute the length of a latus rectum of the ellipses in Problem 3.

5. Write the equations of the following hyperbolas:

(*a*) Center at (2, 1), transverse diameter = 5, conjugate diameter = 2; the transverse axis is horizontal.

(*b*) Center at (−2, 3), transverse diameter = 3, conjugate diameter = 1; the transverse axis vertical.

(*c*) Center at (−1, −4), transverse diameter = 4, conjugate diameter = 2.

6. Locate the foci and compute the length of a latus rectum for each of the hyperbolas in Problem 5.

7. Prove that the distance of any point of the rectangular hyperbola from the center is a mean proportional between the distances from this point to the foci of the hyperbola.

8. Find the equations of the following hyperbolas

(*a*) Transverse diameter $= 8$, foci $(-2, 0)$ and $(10, 0)$

Ans $3(x - 4)^2 - y^2 = 27$.

(*b*) Conjugate diameter $= 2$, foci $(0, 0)$ and $(-4, 0)$

(*c*) Transverse diameter $= 4$, foci $(1, 3)$ and $(-4, 3)$

9. Find the equation of the hyperbola whose eccentricity is 3 and whose latus rectum is 32 cm in length

10. Prove the following properties of hyperbolas

(*a*) The distance from an asymptote of a hyperbola to either focus is numerically equal to b

(*b*) The product of the distances from the asymptotes to any point on the hyperbola is constant

(*c*) The point of contact of a tangent to a hyperbola is midway between the points in which the tangent meets the asymptotes

(*d*) The area of the triangle formed by a tangent to a hyperbola and the asymptotes is constant

SUMMARY

Hyperbola.
  Axes
    Transverse.
    Conjugate
  Center, vertices
  Foci, focal radii, eccentricity
  Type form of equation
  Equation of the tangent at a given point
  Equation of the tangents with a given slope
    Conditions for reality
  Asymptotes
  Conjugate diameters, parallel chords
  The string property of the hyperbola
  The reflection property of the hyperbola
    Construction of the tangent at a given point
  The rectangular hyperbola

The ellipse and hyperbola with center at (h, k)
  Type forms of equations
  Equations of tangents at given points.
  The latus rectum.

CHAPTER VIII

THE PARABOLA. CONIC SECTIONS

In Chap. XII of the first volume of this series a study was made of the quadratic function $y = ax^2 + bx + c$, whose graph is a parabola with a vertical line as axis of symmetry. In this chapter we shall study the forms of equations whose graphs are parabolas with horizontal axes. This will be followed by some general definitions of conics and the proof that the type of a conic is determinable from certain analytical criteria.

48. The parabola. For greater convenience we make a change in notation; instead of designating parabolas with vertical axis and with vertex at (0, 0) and at (h, k) respectively by $y = ax^2$ and $y - k = a(x - h)^2$ let us use

$$4py = x^2 \tag{1}$$

and

$$4p(y - k) = (x - h)^2 \tag{2}$$

as type forms for the parabolas named. It is clear that this entails no loss of generality, since we are merely using the symbol p to replace $1/4a$.

In both cases the parabola is concave upward if p is positive and concave downward if p is negative. The respective derivatives are $\frac{x}{2p}$ and $\frac{x - h}{2p}$.

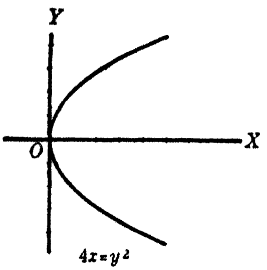
Let the variables x and y and the parameters h and k be interchanged in the two equations; *i.e.*, consider the two double-valued functions of x defined by

$$y^2 = 4px \tag{3}$$

and

$$(y - k)^2 = 4p(x - h). \tag{4}$$

The graphs will again be parabolas; the axis of (3) is the x-axis, the axis of (4) is the line $y = k$; the respective vertices are at (0, 0) and (h, k); the derivatives are $\frac{2p}{y}$ and $\frac{2p}{y - k}$. It is clear that the derivatives cannot be expressed as functions of x alone without the use of radicals, since for each x there are two points on the curve; hence there must be two values of the derivative in general for a given x. The equations show that for a given y there is only one x; hence the derivative is expressible as a single-valued function of y.



$4x = y^2$

FIG 38

The graph of $4px = y^2 (p = 1)$ is given in Fig. 38.

The equation of the line which is tangent to the graph of equation (3) at $P_1(x_1, y_1)$ is

$$y_1 y = 2p(x + x_1). \qquad (5)$$

The x-intercept of this line is $x = -x_1$, an equality which says that *the abscissa of the x-intercept of a line which is tangent to the parabola whose equation is $y^2 = 4px$ is the negative of the abscissa of the point of tangency.*

The student should make the corresponding statement about the tangent to the parabolas $(y - k)^2 = 4p(x - h)$ and $4p(y - k) = (x - h)^2$ and should show how this property furnishes a simple and accurate *method of constructing with ruler and compasses a tangent to a given parabola at any point.*

49. The reflection property of the parabola. In Fig. 39, TO is equal to the abscissa of P_1 and, consequently, P_1T is tangent to $y^2 = 4px$ at P_1; the line P_1R is drawn parallel to OX; the inclination of P_1T is denoted by τ; hence the angle RP_1S is equal to τ. Let the angle TP_1F be constructed equal to τ; then a ray of light RP_1 is reflected to F. We shall compute the abscissa of F and show that it does not depend on the position of P_1; *i.e., all rays which are parallel to the axis of the parabola are reflected to a single point F on the axis.* A second way of expressing this important property of the point F is this: the rays of light from a source at F will be reflected by the parabola in lines parallel to the axis of the parabola. A similar statement is true for a source of sound at F. To the point F is given the name **focus.**

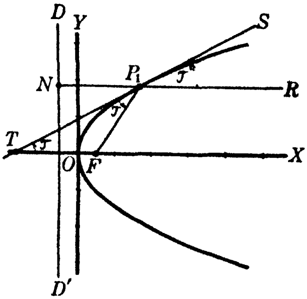


FIG. 39.

This property is utilized in the construction of reflecting telescopes. The mirror has the shape of a paraboloid of revolution; *i.e.*, every plane section through the axis of the telescope cuts the mirror in a parabola with the same point F as the focus. The rays of light from a distant star are essentially parallel; hence they are all reflected to the point F.

Proof of the reflection property. The inclination of FP_1 is equal to 2τ; hence the slope of FP_1 and the equation of FP_1 can be computed in terms of the derivative of y with respect to x. We have

$$\tan 2\tau = \frac{2 \tan \tau}{1 - \tan^2 \tau} = \frac{4p/y_1}{1 - \dfrac{4p^2}{y_1^2}} = \frac{4py_1}{y_1^2 - 4p^2} = \frac{y_1}{x_1 - p}.$$

The equation of FP_1 is

$$y - y_1 = \frac{y_1}{x_1 - p}(x - x_1).$$

The abscissa of F is found by setting $y = 0$ in the last equation and solving for x; the result is $x = p =$ constant. Hence F is a *fixed* point on the axis of the parabola. *q.e.d.*

Note 1. The distance from the vertex to the focus of a parabola is one-fourth of the constant in the equation $y^2 = 4px$.

Note 2. The length of FP_1 = the length of $TF = x_1 + p$; *i.e.*, the length of the focal radius to any point P is equal to $p + x$ where x is the abscissa of P.

Let the line $D'D$ whose equation is $x = -p$ be constructed (in Fig. 39, $p = \frac{1}{2}$) and let NP_1 be perpendicular to this line which is called the **directrix** of the parabola. Then $NP_1 = p + x_1 = FP_1$. Therefore, *an arbitrary point P on a parabola is equidistant from the focus and from the directrix of the parabola.*

EXERCISE XXXVII

1. Find the equations of the tangents to the following parabolas at the points indicated. Draw the graphs. Locate the foci and write the equations of the directrices.

(*a*) $y^2 = 4x$, $(1, -2)$; (*b*) $y^2 = -2x$, $(-2, 2)$;
(*c*) $y^2 = 8x$, $(2, 4)$; (*d*) $y^2 = -9x$, $(-1, -3)$;
(*e*) $4y = x^2$, $(2, 1)$; (*f*) $-16y = x^2$, $(4, -1)$.

2. Compute the length of the latus rectum of each of the parabolas in Problem 1.

3. Compute the distances of the given points from the respective foci.

4. Show that the coordinates of the focus of the parabola whose equation is $4p(x - h) = (y - k)^2$ are $(h + p, k)$ and the equation of the directrix is $x = h - p$.

5. Find the coordinates of the focus and the equation of the directrix of the parabola whose equation is $4p(y - k) = (x - h)^2$.

6. Show that the equation of the tangent at P_1 to the parabola in Problem 4 is

$$y - y_1 = \frac{2px_1}{y_1 - k}(x - x_1).$$

7. Find the equations of the tangents to the following parabolas at the points indicated. Draw the graphs. Locate the foci and write the equations of the directrices.

(a) $8(y - 3) = (x + 1)^2$, $(-5, 5)$; (b) $(y + 1) = x^2$, $(-2, 3)$;
(c) $-2y = (x + 2)^2$, $(2, -8)$; (d) $4(x + 3) = (y - 5)^2$, $(-2, 3)$.

8. Find the lengths of the focal radii of the points in the last problem.

50. Directrices of ellipse and hyperbola. In the last section it was shown that, associated with the focus of a parabola is a line, the directrix, which is perpendicular to the axis of the parabola and is $2p$ units from the focus.

The question arises as to whether there are directrices associated with the foci of the ellipse and of the hyperbola. It can be shown that there are such lines and that they have the property that *the ratio of the distance of a point P on the curve from a focus to its distance from the corresponding directrix is e, the eccentricity.* Since for the ellipse e is less than unity and for the hyperbola e is greater than unity, it follows that a point on an ellipse lies nearer to the focus than to the corresponding directrix, while for a hyperbola the contrary statement holds.

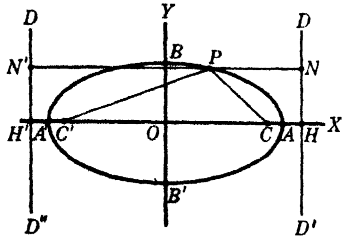


FIG. 40.

The position of the directrix which is associated with the focus $C(c, 0)$ of the ellipse in Fig. 40 will be ascertained in three steps.

1. The point H on the x-axis is located to satisfy $CA/AH = e$.

2. A line is drawn through H perpendicular to the major axis of the ellipse. This line is $D'D$ and is called the directrix.

3. It is proved that $CP/PN = e$, where P is any point on the ellipse and PN is the distance from P to $D'D$.

Proof. 1. The lengths of CA and AH are, respectively, $a - c = a - ae$ and $x - a$. The equation $\frac{CA}{AH} = e$ becomes $\frac{a - ae}{x - a} = e$, from which $x = \frac{a}{e}$.

2. The equation of $D'D$ is $x = a/e$.

3. Let $P(x, y)$ be any point on the ellipse Then $CP = a - ex$ and $PN = a/e - x$. The value of CP/PN is thus equal to e. *q.e.d.*

Associated with the focus C' $(-c, 0)$ in Fig. 40 is the directrix $D''D$ whose equation is $x = -a/e$. The student should prove that $C'P/N'P = e$.

In the same way *the two directrices of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *are the two lines* $x = \frac{a}{e}$ *and* $x = \frac{-a}{e}$. Since for the hyperbola $e > 1$ the number a/e is smaller than a, the directrices of the hyperbola lie between the center and the respective vertices. The proof that $CP/PN = C'P/N'P = e$, where N and N' are defined as in Fig. 40, is made as in the case of the ellipse.

In both of these cases we have chosen a simple type form of equation for the curve under consideration. If the center is not at the origin or if the major axis of the ellipse is vertical, the analytical statement of the equations of the directrices will change. In all cases the student should remember the positions of foci and directrices by *distances from the center or from the vertex*, not by coordinates and equations. It will then be easy to sketch the directrices of an ellipse, hyperbola, or parabola, regardless of the orientation of the curve with respect to the axes of coordinates.

EXERCISE XXXVIII

1. Sketch the following curves showing foci and directrices. Write the coordinates of the foci and the equations of the directrices.

(*a*) $16x^2 + 9y^2 = 144$;

(*b*) $x^2 + 2x + 4y^2 - 4y = 18$;

(*c*) $16x^2 + 25y^2 = 400$;

(*d*) $9x^2 - 36x + 25y^2 + 50y = 164$;

(*e*) $16x^2 - 9y^2 = 144$;

(*f*) $16x^2 - 25y^2 = -400$;

(*g*) $x^2 + 2x - 4y^2 - 4y = 16$;

(*h*) $9x^2 - 36x - 25y^2 - 50y = 214$;

(*i*) $4x^2 + 4y^2 = 25$.

51. Conic sections. The relations of ellipse, parabola, and hyperbola to foci and directrices which have been developed in the last sections suggest that the three curves might have been defined in the beginning by means of this relationship; *viz.*, *the locus of the point P which moves so that CP/NP = e, (e, a positive number), where C is a fixed point and NP is the numerical value of the distance from P to a fixed line, is an ellipse, parabola, or hyperbola according as e, is less than, equal to, or greater than unity.*

Let there be given a fixed point C and a fixed line $D'D$ (Fig. 41). An equation containing e will be derived which, on varying e, represents all of the curves mentioned in the last paragraph.

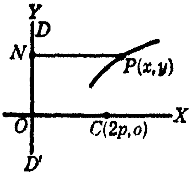


FIG 41

The fixed line $D'D$ is taken as the y-axis of coordinates; the line through the fixed point C perpendicular to $D'D$ is taken as the x-axis. The length of OC is denoted by $2p$. Then the length of CP is $\sqrt{(x-2p)^2+y^2}$ and the length of NP is x. Since P is a point whose coordinates satisfy the condition $CP/NP = e$ or $CP = e \cdot NP$, the locus of P is the curve whose equation is

$$\sqrt{(x-2p)^2+y^2} = ex,$$

or

$$(x-2p)^2+y^2 = e^2x^2,$$

or, finally,

$$x^2(1-e^2) - 4px + y^2 + 4p^2 = 0. \tag{6}$$

For $e = 1$ it is clear that equation (6) is the equation of a parabola whose vertex is at the point whose coordinates are $(p, 0)$, and whose focus and directrix are C and $D'D$, respectively.

For $e \neq 1$, equation (6) may, by completing the square of the terms in x be put into the form

$$\left(x - \frac{2p}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{4p^2e^2}{(1-e^2)^2}. \tag{7}$$

From equation (7) the following conclusions are drawn:

1. When e is less than 1 and not zero, the graph of the equation is an ellipse whose center is on the x-axis.

2. When e is greater than 1, the coefficient of y^2 is negative and, since the right member is always positive, the graph of the equation is a hyperbola.

In both cases C is a focus and $D'D$ is the associated directrix of the curve.

3. When e approaches zero, equation (6) approaches $(x-2p)^2 + y^2 = 0$, an equation which is satisfied by only one real point,

$(2p, 0)$. This equation is called the equation of the **null-circle**;* it is the limiting case of the ellipse. An ellipse with a very small eccentricity differs little from a circle. An ellipse with eccentricity almost equal to unity is narrow in comparison with its length. The limiting case is the parabola for which the eccentricity is unity.

Sections of a cone. *Parabolas, circles, ellipses, and hyperbolas may be obtained geometrically as plane sections of a right circular cone with two nappes.* Let Q be the vertex of such a cone and let the axis of the cone be denoted by $R'R$ (Fig. 42). The angle α whose tangent is TS/TQ is one-half of the **vertex angle** of the cone. We tabulate the various curves, called **conic sections**, or merely **conics**, which are obtained as sections of the cone with a plane.

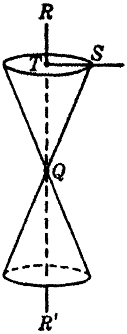


FIG. 42.

The plane	The conic section	Degenerate cases
Is perpendicular to $R'R$	Circle	Null-circle or point-circle if the plane passes through Q
Makes an acute angle greater than α with $R'R$	Ellipse	Point-ellipse if the plane passes through Q
Makes an angle equal to α with $R'R$	Parabola	Element of the cone counted twice, *i.e.*, a double line, if the plane passes through Q
Makes an angle less than α with $R'R$	Hyperbola	Two intersecting lines if the plane passes through Q

The four curves which are named in the middle column of the above table are called **proper conic sections** and also **non-degenerate conic sections**; the **degenerate conic sections** are in the last column. With suitable choice of a coordinate system in the intersecting plane, the equations of the degenerate conics are as follows:

Point-circle · · · $x^2 + y^2 = 0$,
Point-ellipse · · · $ax^2 + by^2 = 0$, $(a \neq b;$ a and b both positive),
Double line · · · $y^2 = 0$,

---

* The null-circle is the only circle whose equation can be referred to the frame of reference which we are using here, *i.e.*, a frame in which the y-axis is directrix.

Two intersecting lines . . . $ax^2 - by^2 = 0$, (a and b both positive),
Two parallel lines . . . $x^2 - a^2 = 0$.

The last is a limiting case when Q moves off to infinity and the cone becomes a cylinder.

52. Conics; analytical criteria. In the last section we established geometrical criteria by means of which we may determine the nature or type of a conic section; the type of the conic depends on the angle between the axis and the intersecting plane, the conic is degenerate if the plane passes through Q, otherwise it is a non-degenerate or proper conic.

In this section will be established for certain cases analytical criteria for determining whether or not a conic is degenerate. Later (Chap. IX) more general criteria can be developed. We shall there find an analytical condition for degeneracy and another condition by use of which it can be told by inspection whether a conic under discussion is elliptic, parabolic, or hyperbolic in type.

Theorem. *The equation in rectangular coordinates of any (proper or degenerate) conic is an equation of the second degree and the graph of any second-degree equation in x and y is a conic.*

The first statement in the theorem needs no proof. From time to time we have studied the equations and constructed the graphs of the various types of conics. In all cases the frame of reference has been so chosen that an axis of the (non-degenerate) conic has been coincident with or parallel to a coordinate axis. When this is the situation we can say that a real non-degenerate conic section has an equation of the type of (8); *viz.*,

$$ax^2 + by^2 + 2gx + 2fy + c = 0, \tag{8}$$

where a and b are not simultaneously zero. The equations of the degenerate conic sections are tabulated at the end of Sec. 52; each of them is included among the totality of equations given in (8).

It will be shown in Secs. 55 to 57 that when the coordinate axes are oblique to the axis or axes of the conic, the equation of the latter is of the form of equation (9); *viz.*,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \tag{9}$$

In Sec. 57 will be presented a method of rotating the coordinate axes until they take a position in which one at least is parallel to an axis of the conic. The effect on equation (9) is to make the xy-term disappear, (its coefficient becomes zero), leaving an equation of the form of (8). Hence equation (9) cannot represent any type of conic which is not representable by equation (8).

To ascertain the type of conic represented by equation (8), we complete the square of the terms in x, if a is not zero, and of the terms in y, if b is not zero. One of the three following forms will be obtained.

$$\begin{aligned} &(a)\ a(x-h)^2 + b(y-k)^2 = K, \\ &(b)\ a(x-h)^2 + 2fy \qquad\quad = K, \\ &(c)\ 2gx \qquad\qquad + b(y-k)^2 = K. \end{aligned} \tag{10}$$

The symbols h, k, f, g, K may have any real values, zero included; the symbols a and b have any real values other than zero. It is evident that 10_b and 10_c are of the same type. The table which follows exhibits the various cases which may arise.

Equation		$K \neq 0$	$K = 0$
10_a	a and b have like signs	Ellipse*	Point-ellipse
	a and b have unlike signs	Hyperbola	2 intersecting lines
	$a = b$	Circle*	Point-circle
10_b	$f \neq 0$	Parabola	Parabola
	$f = 0$	2 parallel lines	2 coincident lines

* When a and K have like signs, the curve is real, otherwise, it is imaginary

EXERCISE XXXIX

Reduce each of the following equations to one of the type forms in equations (10). Interpret.

(a) $3y^2 + 16x - 6y - 1 = 0$;
(b) $5x^2 + 30x - 2y + 45 = 0$;
(c) $9x^2 - y^2 - 8y - 16 = 0$;
(d) $16x^2 - 9y^2 = 0$;
(e) $16x^2 - 9y^2 = -25$;
(f) $x^2 + 9y^2 + 2x + 6y + 9 = 0$;
(g) $5x^2 + 9y^2 + 10x + 6y + 8 = 0$;
(h) $x^2 - 4x + y^2 + 8y + 20 = 0$.

53. Poles and polars with respect to a conic. When the various conics have been studied in type form, it has been shown that the equation of the tangent at the point $P'(x', y')$ is obtained from the equation of the conic by replacing x^2 by $x'x$, y^2 by $y'y$, x by $\frac{1}{2}(x + x')$, and y by $\frac{1}{2}(y + y')$. It will now be shown that this replacement gives the tangent to the general conic at P' if in addition we replace xy by $\frac{1}{2}(y'x + x'y)$.

Proof. Let the equation of the conic be $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. When this equation is differentiated with respect to x and solved for D_xy the result is

$$D_xy = -\frac{ax + hy + g}{hx + by + f}.$$

The equation of the tangent at P' is

$$y - y' = \frac{-ax' - hy' - g}{hx' + by' + f}(x - x'),$$

which, after clearing of fractions and rearranging terms, becomes

$$ax'x + h(y'x + x'y) + by'y + gx + fy = ax'^2 + 2hx'y' + by'^2 + gx' + fy'.$$

Since P' is on the conic its coordinates satisfy the given equation; hence the right member of the last equation can be replaced by $-gx' - fy' - c$. When these terms are transposed to the left member the equation of the tangent at P' may be written

$$\mathbf{ax'x + h(y'x + x'y) + by'y + g(x + x') + f(y + y') + c = 0.} \quad (11)$$

q.e.d.

Let $P'(x', y')$ be a point which is not on the given conic and consider equation (11) in which the coordinates of P' appear. It is the equation of a line which is called the **polar of P' with respect to the conic.** The point P' is called the **pole of the line with respect to the conic.**

Thus, when a conic is given, for every point in the plane (except the center of the conic) there is a polar line and for every line in the plane there is a pole. Furthermore, if P' is not on the conic it is not on its polar line. This is easily seen by substituting the coordinates of P' in equation (11); the coordinates do not satisfy the equation. If P' is on the conic, its polar line is the tangent to the conic at P'.

54. Properties of poles and polars. 1. If P'' is on the polar of P' with respect to a conic, then P' is on the polar of P'' with respect to the conic.

Proof. If P'' is on the polar of P', equation (11) is satisfied when x and y are replaced by x'' and y''; *i.e.*, $ax'x'' + h(y'x'' + y''x') + \cdots = 0$. Now write out the polar of P''; the first terms are $ax''x + h(y''x + x''y) + \cdots = 0$.

Replace x and y by x' and y' and the result is precisely that which was obtained before; *viz.*, $ax''x' + h(y''x' + y'x'') + \cdots = 0$. *q.e.d.*

2. If P' is outside of the conic, its polar cuts the conic in two real points each of which has the property that when it is joined to P' the line thus determined is tangent to the conic.

Proof. From any point P' we know that two real tangents can be drawn to the conic. Denote the points of tangency by Q and R. Then the tangent at Q is the polar of Q with respect to the conic and the tangent at R is the polar by R with respect to the conic. But P' is on the polar of Q and P' is on the polar of R; consequently, Q and R are on the polar of P'. The real points Q and R then are the points of intersection of the polar of P' with respect to the conic, as was asserted. *q.e.d.*

3. If a line cuts the conic in two real points Q and R, the pole of the line is outside the conic and is the point of intersection of the tangents to the conic at Q and R.

The proof is obvious, as is the proof of the next property which is the converse of item 3.

4. If a line cuts the conic in two imaginary points, the pole of the line is inside of the conic.

55. Construction of conics with straight edge and compasses. The common geometric origin of the conics as sections of a right circular cone and the fact that a single equation, *viz.*,

$$x^2(1 - e^2) - 4px + y^2 + 4p^2 = 0,$$

can represent any one of the three types of conics, depending on the value of e, suggests that there may be a method of constructing points on conics by use of ruler and compasses, *i.e.*, without computing values. That such is the case is now shown.

The parabola. Take as directrix the line $D'D$ and as focus the point C in Fig. 43. Through C draw OX perpendicular to $D'D$. Draw a number of lines perpendicular to OX cutting OX at M_1, M_2, . . . With C as a center and OM_1 as a radius describe arcs cutting the line through M_1 in two points P_1 and P_1'; with C as a center and OM_2 as a radius describe arcs cutting the line through M_2 in two points P_2 and P_2'. The points P_1, P_1', P_2, P_2' are points on the parabola. Similarly we may construct as many more points as we wish. In particular, the vertex of the parabola is midway between O and C.

When sufficient points have been found to determine the shape of the parabola, a smooth curve is drawn through them.

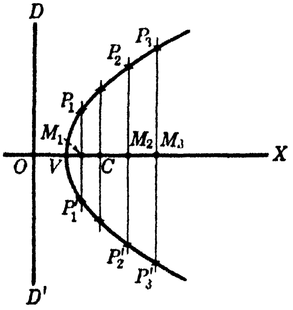


FIG. 43

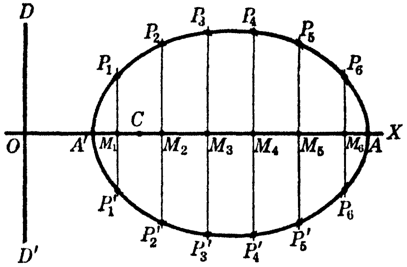


FIG. 44

The ellipse. Take the directrix $D'D$ and the focus C as in Fig. 44. Through C draw OX perpendicular to $D'D$. We wish to construct points on an ellipse whose eccentricity is e (< 1). (In the figure e is $\frac{2}{3}$.) There will be a point A' between C and O on OX for which $CA'/A'O = e$ and a point A on OX to the right of C for which $CA/OA = e$. These two points can be constructed by methods of elementary geometry; they are the vertices of the required ellipse. At any point M between A' and A draw a line perpendicular to OX. Set the compasses so that their spread is $e \cdot OM$. With this radius and C as center describe arcs cutting the vertical line through M in P and P'. These points satisfy the relation $CP/x = e$ and $CP'/x = e$ and they are therefore points on the ellipse, since the directrix has been chosen as the y-axis. As many points may be constructed as are needed to determine the shape of the ellipse and a smooth curve drawn through them.

The hyperbola. Take the directrix $D'D$ and the focus C as in Fig. 45. Through C draw OX perpendicular to $D'D$. We wish to construct points on a hyperbola whose eccentricity is e (> 1). (In the figure e is $\frac{5}{3}$.) There will be a point A on OX between O and C for which $CA/OA = \frac{5}{3}$ and a point A' to the left of $D'D$ for which $CA'/OA' = \frac{5}{3}$. At any point M *to the right of* A *on*

OX or *to the left of* A' *on* OX erect a line perpendicular to OX. Set the compasses so that their spread is $e \cdot OM$. With this radius and C as center describe arcs cutting the vertical line through M in P and P'. These points are points of the hyperbola. As many points may be constructed as are needed to determine the shape of the hyperbola and a smooth curve drawn through them.

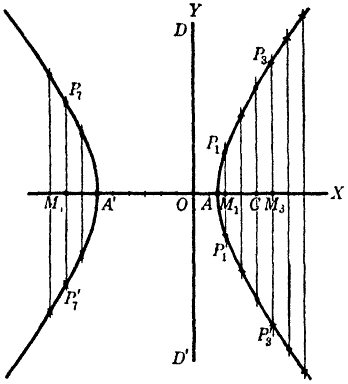


Fig. 45.

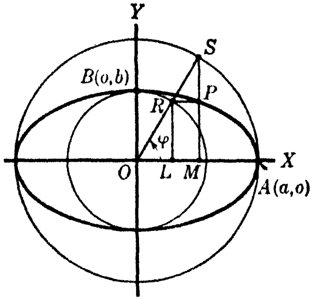


Fig. 46

The ellipse, a second construction. Let the circles in Fig. 46 be graphs of the two equations

$$x^2 + y^2 = a^2,$$
$$x^2 + y^2 = b^2,$$

where $a > b$. Draw any radius ORS from O and denote its inclination by φ. Through R draw RL perpendicular to OX and RP parallel to OX meeting the vertical line MS at P. It can be shown that *the locus of* P *as* φ *varies is an ellipse* with semi-major axis a and semi-minor axis b. As many points as are desired may be constructed and a smooth curve drawn through them.

Proof. We have $OM/OS = \cos \varphi$ and $LR/OR = \sin \varphi$. But $OM = x =$ abscissa of P and $LR = MP = y =$ ordinate of P. Therefore we have the following relations:

$$\mathbf{x = a \cos \varphi, \; y = b \sin \varphi,} \tag{12}$$

from which

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad q.e.d.$$

The circles in Fig. 46 are called the **major auxiliary circle** *and the* **minor auxiliary circle** *of the ellipse.* The angle φ which determines the position of the point P is called the **eccentric angle** φ. The equations (12) give a parametric representation of the ellipse in terms of the eccentric angle.

A second construction of the tangent to an ellipse at a given point. To construct the tangent at P to the ellipse in Fig. 46 draw at S the tangent to the major auxiliary circle and denote the point where this line crosses the x-axis by T. Join PT. The line PT is tangent to the ellipse at P.

Proof. The equation of the tangent to the ellipse at P is $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$ where x_1, y_1 are the coordinates of P. The x-intercept of this line is a^2/x_1. Let the coordinates of S be x_1, Y. The equation of the tangent at S is $x_1x + Yy = a^2$; its x-intercept is a^2/x_1. *q.e.d.*

The hyperbola, a second construction. Let the circles in Fig. 47 be graphs of the two equations $x^2 + y^2 = a^2$, $x^2 + y^2 = b^2 (a > b)$. At B draw BL tangent to the smaller circle. Draw any radius OS from O and denote its inclination by φ. At S draw SM perpendicular to OS, *i.e.*, tangent to the circle whose radius is a. At M erect MK perpendicular to OX. Through the point R which is the intersection of the radius OS with the tangent BL draw a horizontal line meeting MK at P. The locus of the point P as φ varies in a hyperbola.

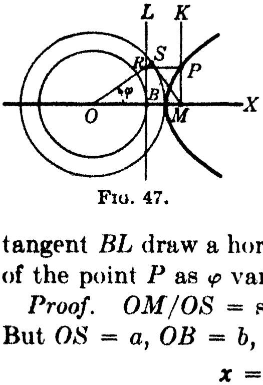


FIG. 47.

Proof. $OM/OS = \sec\varphi$ and $MP/OB = BR/OB = \tan\varphi$. But $OS = a$, $OB = b$, $OM = x$, $MP = y$. Therefore we have

$$\boldsymbol{x = a \sec\varphi,\ y = b \tan\varphi,} \tag{13}$$

from which

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad q.e.d.$$

The circles in the figure are **auxiliary circles** of the hyperbola; the angle φ is called the **eccentric angle.** Equations (13) are parametric equations of the hyperbola.

EXERCISE XL

1. Use the method of Fig. 43 to construct a parabola.

2. Use the method of Fig. 44 to construct 12 points of an ellipse whose eccentricity is $\frac{5}{6}$.

3. Use the method of Fig 45 to construct 12 points of a hyperbola whose eccentricity is $^6/_5$.

4. Use the method of Fig 46 to construct 12 points of an ellipse whose semi-axes have the respective values $a = 10$, $b = 6$

5. Use the method of Fig. 47 to construct 12 points of a hyperbola whose transverse semi-diameter and conjugate semi-diameter have the values $a = 6$, $b = 3$.

6. Use the same method for the hyperbola for which $a = 3$, $b = 6$ ($2a$ = transverse diameter).

7. Construct the tangent to the ellipse in Problem 4 at the point $(-8, {}^{18}/_5)$ Make two different constructions of this tangent

SUMMARY

- The parabola.
  - Type forms of equation.
    - With vertex at the origin.
    - With vertex at (h, k).
  - Equation of the tangent at a given point.
    - Construction of the tangent at a given point.
  - Reflection property.
    - A second construction for the tangent
    - Directrix, focus.
      - Directrices of ellipses and hyperbolas.

- Conic sections.
  - Geometrical generation.
  - Geometrical criteria
    - Proper or degenerate.
    - Type.
  - Analytical criteria.

- Geometrical constructions.
  - The parabola.
  - The ellipse, two constructions.
  - The hyperbola, two constructions.
  - Tangent to the ellipse at a given point.

CHAPTER IX

TRANSFORMATION OF COORDINATES

It is frequently desirable to change the position of the frame of reference. Generally this is done in order to obtain a simplification in the analytical representation of a curve. In Sec. 13, in order to compute the successive figures of an irrational root of an equation $f(x) = 0$, the roots were diminished by a certain number, say k, in order to obtain a new equation with a root near zero. A geometrical interpretation of this process is exhibited in Fig. 15, in which the curve is **translated** two units to the left. The same result is obtained by moving the y-axis two units to the right as in Fig. 48 where the new y-axis is indicated by $O'Y'$.

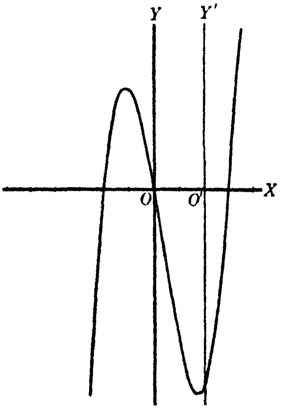


FIG. 48

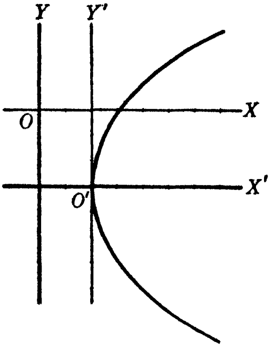


FIG. 49.

Again, the equation of the parabola in Fig. 49 is $y^2 + 6y - 8x + 25 = 0$ when it is referred to the axes OX and OY. The equation may be written $(y + 3)^2 = 8(x - 2)$, a form which shows that the coordinates of the vertex are $(2, -3)$.

Let a new pair of axes, $O'X'$ and $O'Y'$ be taken with O' at the vertex of the parabola, and let x' and y' be the symbols used to denote the coordinates of a point when it is referred to the new frame of reference. The equation of the parabola is now $y'^2 = 8x'$. The parabola has not been changed in appearance or position but the axes have been translated to the new position. Other instances of change of frame of reference will be given in succeeding sections.

56. Translation of axes. In the last example the x'-axis is parallel to the x-axis, the y'-axis is parallel to the y-axis. Any point P may be thought of as having two pairs of coordinates, the one pair (x, y) giving its position relative to OX and OY, the other pair (x', y') giving its position relative to $O'X'$ and $O'Y'$. We say the coordinates (x, y) are *transformed* into the coordinates (x', y') by the translation of the axes from their original position with the origin of coordinates at O to their new position with the new origin at O'. In general, let the coordinates of O' referred to the original axes be (h, k) (Fig. 50). The two pairs of numbers (x, y) and (x', y') have the relations expressed by the equations

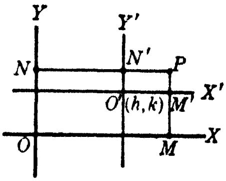


FIG. 50.

$$x' = x - h,$$
$$y' = y - k. \qquad (1)$$

When equations (1) are solved for x and y the result is

$$x = x' + h,$$
$$y = y' + k. \qquad (2)$$

Thus, let us suppose we have a given equation $f(x, y) = 0$ whose graph is a certain curve, C. In the equation we replace x by $x' + h$ and y by $y' + k$ and get a new equation $F(x', y') = 0$, which is the equation of C referred to a new coordinate frame with origin at (h, k) and x'- and y'-axes parallel to the original x- and y-axes. The point h, k may be chosen as desired. In practice, this point is generally so chosen that the equation of the curve is made simpler, as in the case of the parabola whose graph is given in Fig. 49. When the equation contains no terms whose degree is higher than the second degree, the most favorable values of h and k are found by completing the squares in x and in y if square terms in either or both of these terms are present. This is exactly what we have done when we were studying the circle and the conics which did not have the origin of coordinates at the center, or at the vertex in the case of the parabola.

ILLUSTRATIVE EXAMPLE 1. Transform the coordinates in such a way that *the equation* $4x^2 + 16x - 3y^2 + 6y + 1 = 0$ takes one of the type forms of Chap. VIII.

SOLUTION. We have $4(x^2 + 4x) - 3(y^2 - 2y) = -1$. In order to complete the squares we must add 16 and subtract 3; this gives $4(x + 2)^2 - 3(y - 1)^2 = 12$, which is equal to

$$\frac{(x + 2)^2}{3} - \frac{(y - 1)^2}{4} = 1.$$

The transformation is effected by taking $h = -2$ and $k = 1$, *i.e.*, by setting $x = x' - 2$ and $y = y' + 1$; or, what amounts to the same thing, by replacing $x + 2$ by x' and $y - 1$ by y'. This gives as the equation of the curve referred to the new axes

$$\frac{x'^2}{3} - \frac{y'^2}{4} = 1.$$

The student will appreciate the simplification attained by translation of the axes if he attempts to construct by points the graph of the equation in this example without making the suggested transformation of the coordinates.

57. Rotation of axes. A second transformation of coordinates which is used to simplify a given equation is the rotation of the coordinate axes. This is accomplished without changing the position of the origin. The axes in the new positions are called OX' and OY' and the angle through which the positive OX must turn in order to take the position of the positive OX' will be denoted by α. This angle may have any positive or negative value.

If an equation of the second degree contains a term in xy, the conic which is the graph of the equation is not favorably oriented with respect to the axes; *i.e.*, the axes of the conic are not coincident with or parallel to the coordinate axes. If the angle α between these two directions can be found and the axes rotated through this angle, the xy-term disappears (its coefficient becomes zero) and the equation takes one of the type forms which are listed on page 101.

If the origin is not now in the most favorable position the x'- and y'-axes may be translated to new positions with the labels $O''X''$ and $O''Y''$. The type form of the equation in x'', y'' is now such that from it the graph is easily constructed.

Note. It should be kept in mind that transforming the coordinates does not change the appearance of the curve; it changes the equation which defines the curve.

In order to use the theory of rotation of axes to simplify the equations of conics we need to know the following:

1. How the transformed coordinates x' and y' are related to the original coordinates x and y.

2. How to determine the angle α so that the new axes will be parallel to the axes of the conic whose graph is desired.

58. Equations of rotation. In Fig. 51 let OX and OY indicate the positions of a pair of rectangular coordinate axes and let OX' and OY' indicate the positions after rotation through the angle α.

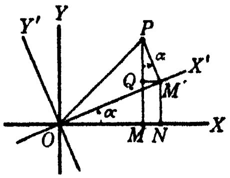


FIG. 51.

A point P in the plane will have two pairs of coordinates, (x, y) and (x', y'), as indicated in the figure; *viz.*,

$$x = OM,\ y = MP,\ x' = OM',\ y' = M'P.$$

The angle $M'PQ$ is equal to the angle α. From the figure we have

$$x = OM = ON - MN;\ y = MP = NM' + QP.$$

Hence,

$$\begin{aligned} x &= x' \cos \alpha - y' \sin \alpha, \\ y &= x' \sin \alpha + y' \cos \alpha. \end{aligned} \tag{3}$$

Multiply the first equation in (3) by $\cos \alpha$, the second equation by $\sin \alpha$, and add. Again, multiply the first equation in (3) by $-\sin \alpha$, the second by $\cos \alpha$, and add. The two results are

$$\begin{aligned} x' &= x \cos \alpha + y \sin \alpha, \\ y' &= -x \sin \alpha + y \cos \alpha. \end{aligned} \tag{4}$$

Equations (4) are called the **inverse** of equations (3); the process of obtaining (4) is called **inverting** the set of equations (3).

By the use of equations (4) the coordinates of a point referred to the x'- and y'-axes can be computed if the x, y coordinates and the angle of rotation are known.

EXERCISE XLI

1. Write equations (4) for $\alpha = 45°$, $\alpha = 120°$, $\alpha = -30°$, $\alpha = 90°$, $\alpha = 60°$.

2. Write equations (4) for $\alpha = \text{arc sin } \frac{1}{3}$, $\alpha = \text{arc tan } (-1)$, $\alpha = \frac{1}{2} \text{ arc tan } 2$, $\alpha = \frac{1}{2} \text{ arc tan } (-3)$.

3. If $\alpha = 30°$, compute the new coordinates, x', y', for the points whose x, y coordinates are as follows: $(4, 3)$, $(-1, 2)$, $(0, 5)$, $(2, -3)$, $(4, 0)$, $(-4, 0)$, $(0, 0)$.

59. The determination of α. Let us consider the general equation of the second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad (5)$$

We shall transform equation (5) into a new equation in x', y' lacking the product term $x'y'$. This will be accomplished by rotating the coordinate axes through an angle α so chosen that the coefficient of the term $x'y'$ in the transformed equation is zero.

When the variables x and y in equation (5) are replaced by $(x' \cos \alpha - y' \sin \alpha)$ and $x' \sin \alpha + y' \cos \alpha$, respectively we get

$$a(x' \cos \alpha - y' \sin \alpha)^2 + 2h(x' \cos \alpha - y' \sin \alpha)\ (x' \sin \alpha + y' \cos \alpha) + b(x' \sin \alpha + y' \cos \alpha)^2 + 2g(x' \cos \alpha - y' \sin \alpha) + 2f(x' \sin \alpha + y' \cos \alpha) + c = 0$$

The new equation is a quadratic equation in x' and y'; let it be denoted by

$$Ax'^2 + 2Hx'y' + By'^2 + 2Gx' + 2Fy' + C = 0. \qquad (6)$$

It can be easily verified that the coefficients A, H, etc., have the following values:

$$\begin{aligned}
A &= a \cos^2 \alpha + 2h \cos \alpha \sin \alpha + b \sin^2 \alpha,\\
H &= (b - a) \cos \alpha \sin \alpha + h(\cos^2 \alpha - \sin^2 \alpha),\\
B &= a \sin^2 \alpha - 2h \cos \alpha \sin \alpha + b \cos^2 \alpha,\\
G &= g \cos \alpha + f \sin \alpha,\\
F &= -g \sin \alpha + f \cos \alpha,\\
C &= c.
\end{aligned}$$

We wish to choose α so that $H = 0$, *i.e.*, so that $h(\cos^2 \alpha - \sin^2 \alpha) + (b - a) \cos \alpha \sin \alpha = 0$. The last equation may be written

$$h \cos 2\alpha = \tfrac{1}{2}(a - b) \sin 2\alpha,$$

and it is satisfied if

$$\frac{2h}{a - b} = \tan 2\alpha. \qquad (7)$$

Hence if we choose $\alpha = \frac{1}{2} \operatorname{arc} \tan \frac{2h}{a - b}$, the new equation has no $x'y'$-term and we proceed as in Sec. 51.

In practice it is not necessary to find the value of α itself; what are needed are $\sin \alpha$ and $\cos \alpha$ which must be used in the equations of rotation. These can be found from the value of $\tan 2\alpha$ without the use of a table of tangents. We need only refer to the double angle formulas;* *viz.*,

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* See Vol. I, p. 102.

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2},$$
$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}. \qquad (8)$$

ILLUSTRATIVE EXAMPLE 1 Rotate the axes of coordinates until they are parallel to the axes of the conic whose equation is $5x^2 - 4xy + 2y^2 = 12$ Sketch the conic

SOLUTION The required angle is determined by

$$\tan 2\alpha = \frac{2h}{a-b} = \frac{-4}{5-2} = -\frac{4}{3}$$

The angle 2α may be an angle of the fourth quadrant or an angle of the second quadrant We may choose either of the two Say we choose $2\alpha = \text{arc tan } -\frac{4}{3}$ *in the second quadrant* then $\cos\alpha$ and $\sin\alpha$ are both positive since α is an angle of the first quadrant Hence

$$\cos 2\alpha = -\frac{3}{5},$$
$$\cos\alpha = +\sqrt{\frac{1-\frac{3}{5}}{2}} = \frac{1}{\sqrt{5}},$$
$$\sin\alpha = +\sqrt{\frac{1+\frac{3}{5}}{2}} = \frac{2}{\sqrt{5}},$$

and the equations of rotation are

$$x = \frac{x' - 2y'}{\sqrt{5}}, \quad y = \frac{2x' + y'}{\sqrt{5}}.$$

Let these values of x and y be substituted in the given equation. The succeeding steps are obvious

$$5x^2 - 4xy + 2y^2 = 12,$$
$$\frac{5(x' - 2y')^2}{5} - \frac{4(x' - 2y')(2x' + y')}{5} + \frac{2(2x' + y')^2}{5} = 12,$$
$$x'^2 + 6y'^2 = 12$$

The graph is an ellipse It is constructed in the following steps

1 Draw the axes OX and OY (Fig 52)

2 Construct angle $XOJ = 2\alpha = \text{arc tan } -\frac{4}{3}$ in standard position in the second quadrant

3 Bisect the angle XOJ This gives the position of OX'.

4 Construct OY' perpendicular to OX' at O

5. Construct the graph of $\frac{x'^2}{12} + \frac{y'^2}{2} = 1$ on the $x'y'$-axes. The graph is shown in the figure

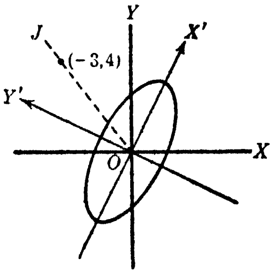

FIG 52.

ILLUSTRATIVE EXAMPLE 2. In the last example *choose 2α in the fourth quadrant* and complete the solution

SOLUTION. If 2α is in the fourth quadrant, α is in the second quadrant, $\cos\alpha$ is negative, and $\sin\alpha$ is positive. Their values are (since $\cos 2\alpha = +\frac{3}{5}$)

$$\cos\alpha = -\sqrt{\frac{1+\frac{3}{5}}{2}} = -\frac{2}{\sqrt{5}},$$

$$\sin\alpha = +\sqrt{\frac{1-\frac{3}{5}}{2}} = +\frac{1}{\sqrt{5}},$$

and the equations of transformation are

$$x = \frac{-2x' - y'}{\sqrt{5}}, \quad y = \frac{x' - 2y'}{\sqrt{5}}.$$

When these values are substituted in the given equations, the simplified result is $6x'^2 + y'^2 = 12$. To make the graph we proceed as before. The angle $XOJ = 2\alpha = \text{arc}\tan -\frac{4}{3}$ is constructed in the fourth quadrant. It is bisected (Fig. 53) by the line OX', the line OY' is drawn perpendicular to OX' and on these axes is constructed the graph of the ellipse

$$\frac{x'^2}{2} + \frac{y'^2}{12} = 1.$$

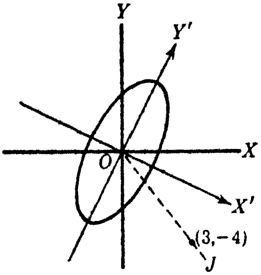

FIG 53

A comparison of the two sketches shows that $+OX'$ in Fig. 52 is $+OY'$ in Fig. 53 and that $+OX'$ in Fig. 53 is $-OY'$ in Fig. 52.

EXERCISE XLII

1. Find a favorable angle of rotation for each of the following. Construct the graphs.

(a) $x^2 + xy + y^2 - 1 = 0$;
(b) $2x^2 - 12xy - 3y^2 + 14 = 0$;
(c) $x^2 - 4xy + 4y^2 - 8 = 0$;
(d) $x^2 - 4xy + 4y^2 - 2y - 3 = 0$.

2. Reduce the following conics to type form

(a) $xy = 4$ *Ans.* $x'^2 - y'^2 = 8$.
(b) $x^2 - 10xy - 40x\sqrt{13} - 8y\sqrt{13} + 25y^2 = 0$. *Ans.* $y'^2 = 8x'$.
(c) $25x^2 + 14xy + 25y^2 = 288$. *Ans.* $16x'^2 + 9y'^2 = 144$.
(d) $3x^2 - 10xy + 3y^2 = 0$. *Ans.* $x'^2 - 4y'^2 = 0$.

60. Invariants under rotation. There are two remarkable facts to be noted in connection with the transformation by rotation. Refer again to the values of A, B, etc., the coefficients of the various terms in the transformed equation (6). It is easy to verify that the following relations hold and are independent of α:

$$\begin{aligned}&(1)\quad A + B = a + b,\\&(2)\quad H^2 - AB = h^2 - ab.\end{aligned}$$

In particular, if α is so chosen that $H = 0$, the above two relations become

$$\begin{aligned}&(1)\quad A + B = a + b,\\&(2)\quad AB = ab - h^2.\end{aligned}$$

From these relations we can tell the type of the conic; *e.g.*, it is called parabolic if it is a proper parabola or if it is a pair of coincident lines. (*Cf.* degenerate cases in the tabulations in Secs. 51 and 52.) It is not necessary to transform the conic to ascertain its type. Consider the numbers A and B, coefficients of x'^2 and y'^2, respectively, in the transformed equation. If these two numbers have unlike signs, the conic is hyperbolic in type; if they have like signs, the conic is elliptic; if one of them is zero, the conic is parabolic. In the three cases, the product AB will be, respectively, negative, positive, zero. Hence, *the type of conic depends on* AB but since $AB = ab - h^2$ we can say

1. *If* $ab - h^2 < 0$, *the conic is hyperbolic in type.*
2. *If* $ab - h^2 > 0$, *the conic is elliptic in type.*
3. *If* $ab - h^2 = 0$, *the conic is parabolic in type.*

It still remains to find under what condition the conic degenerates into one of the three types, two intersecting lines, a point or imaginary ellipse, or two coincident lines. We recall that the second-degree equation which is satisfied by only one real point may be reduced to the type form $ax^2 + by^2 = 0$, with $ab > 0$. This equation may also be looked upon as the equation of the two imaginary lines $x\sqrt{a} + iy\sqrt{b} = 0$ and $x\sqrt{a} - iy\sqrt{b} = 0$.

Hence, with this interpretation of the elliptic type of degenerate conic, we seek the condition that the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ may factor into two linear factors.

If the equation factors, the two factors may be obtained by solving for y by the quadratic formula.* This gives

$$by = -(hx + f) \pm \sqrt{R},$$

where $R = (h^2 - ab)x^2 + 2(hf - bg)x + (f^2 - bc)$. Obviously R must be a perfect square so that its square root may be added and subtracted to $-(hx + f)$ to give linear equations. The condition that R be a perfect square is that its discriminant be zero. This discriminant consists of the factor b which is not zero and the polynomial

---

* The equation can be solved for y if $b \neq 0$, otherwise it can be solved for x if $a \neq 0$; when both a and b are zero if the equation is factorable, the factors can be found by inspection. It can be shown that the condition for factorability is the same as is obtained for the case $b \neq 0$.

$$abc - af^2 - bg^2 - ch^2 + 2fgh$$

which may be written in the determinantal* form:

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

and we have proved that *if the quadratic equation factors, D is zero.*

It can be shown that the value of the corresponding determinant whose elements are the coefficients of the transformed equation (6); *viz.*,

$$\triangle = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

is the same as the value of D. Hence we can add D to our list of **invariants** at the top of page 115. The phrase **invariant under rotation of axes** is used to indicate that these functions of the coefficients are unchanged when the axes are rotated. Geometrically, it means that if the conic is non-degenerate before rotation of axes, it is non-degenerate after the axes have been rotated; if its type is parabolic, hyperbolic, elliptic, rotation does not change this property.

Thus when it is required to find the nature of the conic whose graph is a given equation, the criteria above suffice. Transformation by rotation is necessary only if the graph is to be constructed.

EXERCISE XLIII

1. Find the type of each of the following conics and determine whether it is degenerate or non-degenerate.

(*a*) $25x^2 - 40xy + 16y^2 - 20x + 18y = 0;$
(*b*) $4x^2 + 4xy + y^2 + 4x + 2y - 3 = 0;$
(*c*) $x^2 + 4xy + 4y^2 - 2x - 4y + 1 = 0;$
(*d*) $3x^2 + 4xy + y^2 + 4 = 0;$

---

* This may be taken as a definition of a determinant with three rows and three columns, called a **determinant of third order.**

(e) $xy + 3x - 4y = 8$;
(f) $x^2 - xy + 2y^2 + x + 4y - 2 = 0$;
(g) $y^2 = 4y$.

2. Show that each of the following conics is non-degenerate. Find foci, directrices, eccentricity. Sketch the graphs.

(a) $15x^2 + 6xy + 15y^2 - 3\sqrt{2}x - 3\sqrt{2}y - 34 = 0$;
(b) $x^2 + 2\sqrt{3}xy - y^2 = 18$;
(c) $9x^2 - 24xy + 16y^2 - 20x - 15y = 0$.

SUMMARY

Transformation of coordinates.

Translation.

$$x = x' + h,$$
$$y = y' + k.$$

Rotation.

Equations.

$$x = x' \cos \alpha - y' \sin \alpha,$$
$$y = x' \sin \alpha + y' \cos \alpha,$$
$$\tan 2\alpha = \frac{2h}{a - b}.$$

Invariants.

The conic is degenerate if $D = 0$.

The conic is:

$$\left.\begin{array}{l}\text{Parabolic}\\ \text{Hyperbolic}\\ \text{Elliptic}\end{array}\right\}\text{according as}\left\{\begin{array}{l}ab - h^2 = 0\\ ab - h^2 < 0\\ ab - h^2 > 0\end{array}\right.$$

CHAPTER X

POLAR COORDINATES

Locating points in a plane by means of a system of Cartesian coordinates may be compared to giving directions in a city—a certain number of blocks east, for example, then a certain number of blocks south. There are numerous other ways of setting up a frame of reference in a plane so that, when the two coordinates of a point are given, the point is uniquely determined. In this chapter will be studied the system of **polar coordinates.** Locating points in a plane by this system may be compared to giving directions on a prairie or a desert where a place is located by its distance and direction from a given point.

61. Polar coordinates. Let O be a given point called the **pole** and OX drawn to the right from O be the line, called the **prime direction,** from which directions are to be measured. Choose a unit of length and indicate it on OX. Take P any point in the plane (Fig. 54) and let r denote the length of OP measured in the chosen linear unit and θ the angle (any one of the angles) through which OX may turn in order to take the direction of OP. The two coordinates r and θ serve to locate P without ambiguity.

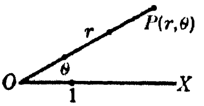

FIG. 54.

To plot points and graphs in polar coordinates the student will find it convenient to have polar coordinate paper. This paper has rays radiating at regular intervals from the pole with a linear scale on the rays and a circular scale to indicate the angles.

The line OP is called the **radius vector** of the point P; θ is the **vectorial angle.**

The student will note that there is only one point with a given pair of values r and θ as coordinates; on the other hand, when a point is given its coordinates are not unique. For example, if the angle θ in Fig. 54 is 30° and the length of OP is 3, the point P may be denoted not only by (3, 30°) but also by (3, −330°), (−3, −150°), (−3, 210°)* and by other pairs of numbers obtained by replacing the above angles by other angles obtained from them by adding multiples of 360°.

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* Some writers prefer not to admit negative values of r, in which case such an equation as $r^2 = \cos 2\theta$ is equivalent to $r = +\sqrt{\cos 2\theta}$ rather than $r =$

EXERCISE XLIV

1. Plot on polar coordinate paper the following points:
$(2, 30°)$, $(4, 150°)$, $(1, 270°)$, $(1, -90°)$, $(-3, 45°)$, $(3, -135°)$, $(-1, 180°)$.
2. Plot the graph of the equation $r = 1$.
3. Plot the graphs of the equations $\theta = 40°$, $\theta = -100°$, $\theta = -140°$.

An equation in r and θ may contain θ algebraically, as $r\theta = 2$, $r = 5\theta^2 - 2\theta$, etc., or it may contain θ in the argument of one or more trigonometric functions, as $r = 3 \sin \theta$, $r^2 = 4 \cos 3\theta$, etc. In the first case the values of θ are to be given in radians.

In constructing graphs of equations $f(r, \theta) = 0$, consideration should be given to the notions of *periodicity* and of *symmetry.* All trigonometric functions are periodic; hence any relation involving trigonometric functions should be examined to see if it has a period. If so, the graph *may* close at a certain point and no new points be obtained by taking values of θ larger than the one which closed the curve. It is not possible to give a rule which shows what values of θ should be included in a table in order that no portion of the curve may be omitted.

We shall now discuss symmetry in a curve in polar coordinates, giving a few of the simplest tests. We may speak of symmetry with respect to the prime direction and of symmetry with respect to the 90° ray, the 180° ray, the 270° ray, etc. However, we shall later superpose our polar coordinate frame of reference on a rectangular frame of reference, allowing the prime direction of the one to coincide with OX of the other and the 90° ray to coincide with OY. Hence, let us at this stage speak of symmetry with respect to the x- and y-axes.

62. Tests of symmetry. Two points which are symmetric with respect to the x-axis, as P_1 and P_2 in Fig. 55, have as polar coordinates (r, θ) and $(r, -\theta)$. *The graph of* $f(r, \theta) = 0$ *is symmetric with respect to the line* OX *if the equation is essentially unchanged* when* θ *is replaced by* $(-\theta)$.

$\pm\sqrt{\cos 2\theta}$. We agree here to use the latter equation and to look upon r as a double-valued function of θ.

* An equation is essentially unchanged if the signs of all terms are changed; this does not change the roots.

Two points which are symmetric with respect to the y-axis, as P_1 and P_1' in Fig. 55, have as polar coordinates (r, θ) and $(r, \pi - \theta)$ or $(-r, -\theta)$. *Hence the graph of* $f(r, \theta) = 0$ *is symmetric with respect to the y-axis if the equation is essentially unchanged when θ is replaced by $\pi - \theta$, or if simultaneously r and θ are replaced by* $-r$ and $-\theta$.

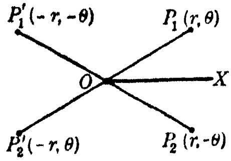


FIG. 55.

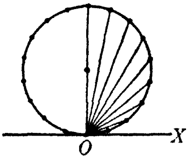
Two points which are symmetric with respect to the origin, as P_2' and P_1, have as polar coordinates (r, θ) and $(-r, \theta)$ or (r, θ) and $(r, \pi + \theta)$. *The graph of* $f(r, \theta) = 0$ *is symmetric with respect to the pole if the equation is of the form* $r^2 = f(\theta)$ *or if the equation is essentially unchanged when θ is replaced by $\pi + \theta$.*

The student should show that the function $r = a \cos 2\theta$ satisfies all of the tests of symmetry given above.

ILLUSTRATIVE EXAMPLE 1. Discuss the function $r = a \sin \theta$ and construct its graph.

DISCUSSION. *Tests for symmetry.* (1) Since $\sin(-\theta) \neq \sin \theta$, the graph is not symmetric with respect to the x-axis. (2) Since $\sin(\pi - \theta) = \sin \theta$, the graph is symmetric with respect to the y-axis. (3) Since $\sin(\pi + \theta) \neq \sin \theta$, the graph is not symmetric with respect to the origin.

We note that $\sin(\pi + \theta) = -\sin \theta$; hence the points obtained by allowing θ to vary from π to 2π will fall on points already obtained; *i.e.*, the graph closes with $\theta = \pi$. In the table of values, it is only necessary to include values of θ from 0° to 90°. Why? The graph is a circle. (See Fig. 56.)



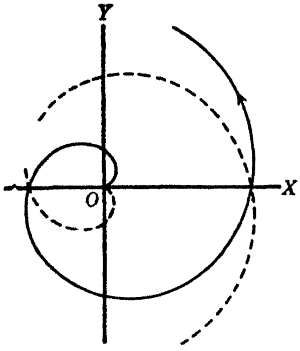
$r = a \sin \theta$
FIG. 56.

θ	$r = a \sin \theta$
0°	0 00
10°	0 17a
20°	0 34a
30°	0 50a
40°	0 64a

θ	$r = a \sin \theta$
50°	0 77a
60°	0 87a
70°	0 94a
80°	0 98a
90°	1 00a

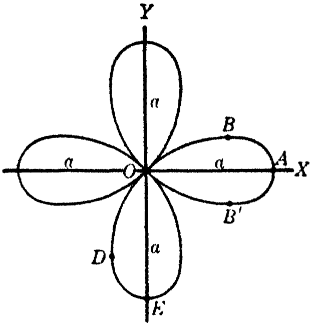
ILLUSTRATIVE EXAMPLE 2. Discuss and construct the graph of the spiral of Archimedes, $r = a\theta$.

DISCUSSION. There is symmetry with respect to the y-axis, since r becomes $(-r)$ when θ becomes $(-\theta)$. As θ increases, r increases. In the graph in Fig 57, the dotted curve gives the function for negative values of θ, the solid curve gives it for positive values of θ.



$r = a\theta$

FIG 57



$r = a \cos 2\theta$

FIG 58.

ILLUSTRATIVE EXAMPLE 3. Discuss and construct the graph of $r = a \cos 2\theta$.

DISCUSSION. We have already found (page 120) that there is symmetry with respect to both axes and with respect to the pole. Hence we need include in the table only values of θ which correspond to points in the first quadrant.

The arc ABO in Fig. 58 is obtained by letting θ vary from 0° to 45°. Hence the arc $AB'O$ corresponds to values of θ from $-45°$ to 0°.

If a second portion of the graph appears in the first quadrant, it will correspond to values of θ from 45° to 90° or from $-90°$ to $-135°$. We find that the values of θ from 45° to 90° correspond to the points on the arc ODE. The entire curve can now be sketched. To get all of the points of the four leaves (it is called the **four-leaved rose**) we may allow θ to vary from 0° to 360° or from $-180°$ to 180°

EXERCISE XLV

Construct the graphs of the following:

(a) $r = a/\theta$ (the hyperbolic spiral);
(b) $r = a \cos \theta$;
(c) $r = a \sin 2\theta$;
(d) $r = a \sin 3\theta$;
(e) $r = a \cos 3\theta$;
(f) $r = a \sin^3 (\theta/3)$;
(g) $r = a \sec 2\theta$;
(h) $r = a (\cos \theta - \sin \theta)$.

63. Relations connecting polar and rectangular coordinates. Let the coordinates of P (Fig. 59) be (r, θ) referred to the pole O and the prime direction OX.

Set up a rectangular coordinate system with O as origin and OX as x-axis. Referred to this system, the coordinates of P are x and y. From the figure it is clear that the two sets of coordinates have the following relations:

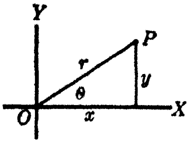


Fig. 59.

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned} \tag{1}$$

The inverse of the relations in equations (1) are

$$\begin{aligned} r &= \sqrt{x^2+y^2}, \\ \theta &= \arc\tan\frac{y}{x}. \end{aligned} \tag{2}$$

An examination of the two sets of equations shows that if the polar coordinates r and θ of a point are known, its abscissa and ordinate are uniquely determined by (1). If, on the other hand, the rectangular coordinates x and y are known, r is uniquely determined by the first but θ is not uniquely determined by the second of equations (2). However, the signs of the given coordinates x and y show the quadrant of θ without ambiguity.

Illustrative Examples. The Cartesian coordinates of the point $(5, \pi/3)$ are $(5/2, 5/2\sqrt{3})$; the polar coordinates of $(-4, 0)$ are $(4, \pi)$.

By the use of equations (1) we may find the polar equation of a given curve whose Cartesian equation is known. The student will be able to verify the correctness of the entries in the following table and to describe each locus.

Cartesian equation	Polar equation
$x = c$	$r\cos\theta = c$
$y = c$	$r\sin\theta = c$
$ax + by = 0$	$\tan\theta = -a/b$
$x^2 + y^2 = a^2$	$r = a$
$(x - a)^2 + y^2 = a^2$	$r = 2a\cos\theta$
$x^2 + (y - a)^2 = a^2$	$r = 2a\sin\theta$
$(x - h)^2 + (y - k)^2 = h^2 + k^2$	$r = 2a\cos(\theta - \gamma)$ (a, γ) are the polar coordinates of (h, k)

Similarly, by the use of equations (2) we may find the Cartesian equation of a given curve whose polar equation is known.

EXERCISE XLVI

1. Find the Cartesian equations of the curves whose equations in polar coordinates are as follows:

(a) $r = a \sin 2\theta$;
(b) $r = a \sec \theta$;
(c) $r^2 - 6r \cos \theta + 5 = 0$;
(d) $r^2 - 2cr \cos(\theta - \gamma) + c^2 = a^2$;
(e) $r \sin \theta = h$;
(f) $r^2 = 25 \sin \theta$;
(g) $r = \sec^2 \frac{\theta}{2}$.

2. Make the graph of the equation $r = a \sin^3 \frac{\theta}{3}$.

64. Conics. Let the focus of a conic be the pole in a system of polar coordinates and let the prime direction be perpendicular to the directrix and directed from $D'D$ toward O (Fig. 60); also let $KO = 2p$. By the definition of a conic, P is a point on a conic whose eccentricity is e if $OP/NP = e$. Now $OP = r$ and $NP = KO + OM = 2p + r \cos \theta$. Hence the equation of the conic is $\dfrac{r}{2p + r \cos \theta} = e$. When this equation is solved for r the result is

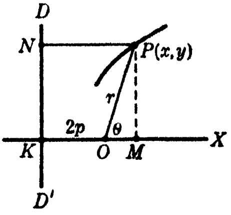


FIG. 60.

$$r = \frac{2pe}{1 - e \cos \theta}. \tag{3}$$

For $e \leqq 1$ the denominator of equation (3) is never negative; hence all values of r are positive. The graph lies to the right of $D'D$ and is an ellipse or a parabola. For $e > 1$, the denominator of equation (3) is negative for $\theta = 0°$ and for values of θ which are numerically less than arc cos $\frac{1}{e}$. These values of θ give points on the branch of the hyperbola which lies to the left of the directrix, while values of θ for which $180° \geqq |\theta| >$ arc cos $\frac{1}{e}$ correspond to points on the branch of the hyperbola which lies to the right of the directrix.

EXERCISE XLVII

1. Sketch the graph of $r = \dfrac{2pe}{1 + e \cos \theta}$ and compare it with the graph of equation (3).

2. Sketch the conics represented by each of the following equations:

(a) $r = \dfrac{4}{1 - \frac{1}{2}\cos\theta}$; (b) $r = \dfrac{4}{1 + \frac{1}{2}\cos\theta}$;

(c) $r = \dfrac{3}{1 - \frac{1}{8}\cos\theta}$; (d) $r = \dfrac{3}{1 - \frac{1}{8}\cos\theta}$;

(e) $r = \dfrac{12}{1 - 4\cos\theta}$; (f) $r = \dfrac{9}{1 - \cos\theta}$;

(g) $r = \dfrac{12}{1 - 4\sin\theta}$; (h) $r = \dfrac{2}{1 - \cos\theta}$;

(i) $r = \dfrac{2}{1 + \cos\theta}$; (j) $r = \dfrac{8}{1 - 3\cos\theta}$;

(k) $r = \dfrac{12}{2 + 4\cos\theta - 3\sin\theta}$; (l) $r = \dfrac{4}{1 + \cos\theta - \sin\theta}$.

3. In Problem 2 (*a*), (*f*), and (*g*), locate the foci, the center, the vertices, and the second directrix of any conic not a parabola. Compute the lengths of the semi-axes and the length of each latus rectum.

65. Special curves. Certain curves which are frequently met in higher mathematics and which are interesting because of their classical origin are more simply defined in polar coordinates than in Cartesian coordinates.

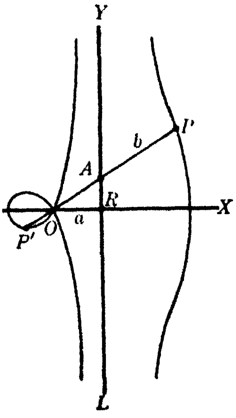


FIG 61
$r = a \sec\theta \pm b$

The conchoid. We have given a fixed line L and a fixed point O which is taken as the pole in a system of polar coordinates. The prime direction OX is perpendicular to L as in Fig. 61, and $OR = a$. The variable line OA whose inclination to OX is θ meets L in A. Lay off $AP = b$ and $AP' = b$. The conchoid is the locus of the two points P and P'. Its equation may be determined as follows: The length of OA is $a \sec\theta$, the length of $OP = r = a \sec\theta + b$ while $OP' = a \sec\theta - b$. Hence the complete curve is given by $\boldsymbol{r = a \sec\theta \pm b}$.

This curve can be used to solve a *famous problem of antiquity, the problem of trisecting an angle.* The ancients thought they ought to find a way of trisecting any given angle by use of straight edge and compasses. It has been proved in modern times that this is not possible, though a very simple construction is possible with the aid of the conchoid. The name conchoid means "mussel-like." It was invented by the Greek mathematician Nicomedes about 200 B.C.

To trisect an angle. Let θ = angle XOP be the given acute angle which is to be trisected (*cf.* Fig. 62) and let R be any point except the vertex on the initial side of θ. Draw the perpendicular to OX at R and denote its intersection with the terminal side of θ by A. With O as the pole and $b = 2OA$ construct the part of the conchoid which corresponds to values of the polar angle from 0° to the given angle θ. Draw AB parallel to OX and cutting the conchoid in B. Then the angle $XOB = \frac{1}{3}\theta$.

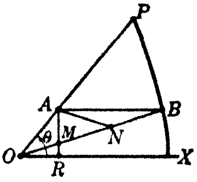


FIG. 62.

Proof. Take N midway between M and B. Since $MB = 2OA$ we have $MN = OA$; also since the angle MAB is a right angle the line AN is equal to the line MN; hence $AN = OA$ and angle NOA = angle ONA = 2 times angle ABN = 2 times angle XOB. *q.e.d.*

The limaçon (snail) of Pascal. Let O be a fixed point on a given circle whose radius is a and let OA be any chord through O (Fig. 63). Lay off $AP = AP' = k$ (constant). The locus of the two points P and P' is the curve which is called the limaçon. Its equation may be found as follows:

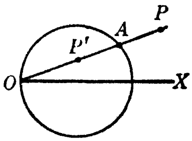


FIG. 63.

$$r = OP = OA + AP = 2a \cos \theta + k,$$
$$r = OP' = OA - P'A = 2a \cos \theta - k.$$

Hence the locus of the two points is given by

$$r = 2a \cos \theta \pm k.$$

There are three cases; *viz.*, $k > 2a$, $k = 2a$, $k < 2a$. The student should plot several of these curves, taking convenient values for a and k. In the case $2a = k$ the equation becomes

$$r = 2a(\cos \theta \pm 1).$$

The graph of $r = 2a(\cos \theta + 1)$ is a heart-shaped figure and is called the **cardioid.**

How do the graphs of $r = 2a(\cos \theta - 1)$ and $r = 2a(1 - \cos \theta)$ differ? Compare them with the graph of $r = 2a(\cos \theta + 1)$.

The cissoid of Diocles. Let O be a fixed point on a circle whose radius is a, and let the prime direction pass through the center of the circle and cut the circle at O and at A. Construct AL tangent to the circle at A.

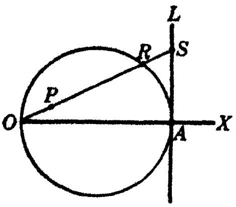


FIG. 64.

Let the variable line through O cut the circle at R and the line AL at S. The point P is chosen so that $OP = RS$. Find the equation of the locus of P.

We have $OS = 2a \sec \theta$ and $OR = 2a \cos \theta$. Then

$$\begin{aligned} r = OP &= OS - PS \\ &= OS - OR = 2a \sec \theta - 2a \cos \theta \end{aligned}$$

$$= 2a(\sec \theta - \cos \theta) = 2a\left(\frac{1 - \cos^2 \theta}{\cos \theta}\right) = \frac{2a \sin^2 \theta}{\cos \theta}.$$

Hence $\boldsymbol{r = 2a \sin \theta \tan \theta}$.

The cissoid may be used to solve the *ancient problem of duplicating the cube, i.e.*, of finding a length whose cube is twice a given cube. This is another problem which cannot be solved with straight edge and compasses.

To duplicate a cube. Construct a cissoid with $a = 1$. At C the center of the circle in Fig. 65 draw a vertical line cutting the circle and the cissoid at D; take $DQ = CD = 1$. The line joining Q with A meets the cissoid at a point which we call B. The perpendicular BM from B to the prime direction has the property that $\overline{MB}^3 = 2\overline{OM}^3$.

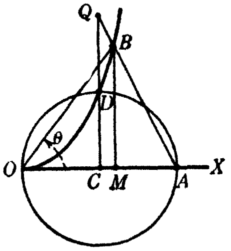


FIG 65.

Proof. We have

(1) $\tan \theta = MB/OM$.

(2) $MA = 2 - OM = 2 - OB \cos \theta = 2 - 2 \sin^2 \theta = 2 \cos^2 \theta$, since $OB = 2 \sin \theta \tan \theta$ from the equation of the cissoid.

(3) $MB/MA = CQ/CA = 2, \therefore MB = 4 \cos^2 \theta$ (from 2).

(4) From (2), $OM = 2 - 2 \cos^2 \theta = 2 \sin^2 \theta$.

(5) Substituting from (2) and (4) in (1) we get $\tan \theta = 2 \cot^2 \theta, \therefore \tan^3 \theta = 2$.

Hence,

$$\overline{MB}^3 = 2\overline{OM}^3, \text{ as was asserted.} \qquad q.e.d.$$

The Cassinian ovals. Let two fixed points whose distance apart is $2a$ be denoted by F_1 and F_2. Find the locus of the point P which moves so that $F_1P \cdot F_2P = b^2$ (b = constant).

Discuss the curve for the three cases $a > b$, $a = b$, $a < b$. When $a = b$ the curve is called the **lemniscate.**

EXERCISE XLVIII

1. Plot the conchoids for which $a = 4$, $b = 6$; $a = 4$, $b = 4$; and $a = 4$, $b = 2$.

2. Plot the limaçons for $a = 6$, $k = 3$; $a = 6$, $k = 6$; and $a = 6$, $k = 8$.

3. Plot the cissoid for $a = 4$.

4. Plot Cassinian ovals for $a = 6$, $b = 4$; $a = 6$, $b = 6$; and $a = 6$, $b = 8$.

5. Find the Cartesian equations for the conchoid, limaçon, cissoid, and Cassinian ovals. (Use a and b or a and k to designate the constants.)

SUMMARY

- Polar coordinates.
  - Pole, prime direction, radius vector, vectorial angle.
  - Symmetry.
    - With respect to the x-axis.
    - With respect to the y-axis.
    - With respect to the pole.
  - Relations between polar and Cartesian coordinates.
  - The equation of a conic in polar coordinates.
  - Special curves.
    - Conchoid.
      - Used in trisecting an angle.
    - Cissoid.
      - Used in duplicating a cube.
    - Limaçon, Cardioid.
    - Ovals of Cassini, Lemniscate.

CHAPTER XI

CERTAIN SPECIAL CURVES. PENCILS OF CURVES

In the first volume of "Elementary Mathematical Analysis" the graphs of all types of elementary single-valued functions were studied; *viz.*,

y = a polynomial function of x,
y = a rational function of x,
y = a trigonometric function of x,
y = a logarithmic function of x,
y = an exponential function of x.

In addition, certain functions with infinitely many values for a given x; *viz.*, y = an inverse trigonometric function, and also the double-valued function whose graph is a circle, were sketched and studied.

In this book we have discussed double-valued algebraic functions of x defined by the general equation of the second degree, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, and in addition a number of curves whose equations are given in polar coordinates.

In this chapter we shall discuss a number of important curves whose equations are most simply given in **parametric form**; *i.e.*, x is a given function of a parameter, say, t or θ, and y is also a given function of the same parameter. A table of values of x and y is constructed by letting the parameter take a succession of admissible values after which the number pairs (x, y) are plotted in an xy-plane where a rectangular frame of reference has been chosen.

A second topic in this chapter deals with systems of curves. For example, the two equations $y = 5x + k$ and $y = mx + 1$ represent, respectively, a pencil of parallel lines and a pencil of lines through a point. In each of these cases there is a single arbitrary constant, and hence a single infinity of lines in the pencil.

We shall study pencils of circles and pencils of conics.

66. Parametric equations. Algebraic curves. Let the values of x and y be given by the two equations

$$x = 2t + 3,$$
$$y = -t + 1.$$

From the appended table the points (x, y) can be plotted. That the graph is a straight line (Fig. 66) is evident from the table, inasmuch as it shows that *both x and y change uniformly with t.*

t	x	y
-3	-3	4
-2	-1	3
-1	1	2
0	3	1
1	5	0
2	7	-1

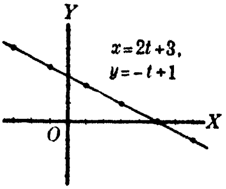


FIG 66

More generally, let the values of x and y be given as linear functions of t by the two equations

$$x = bt + d, \qquad (1)$$
$$y = -at + k.$$

That the graph of y, the function of x defined by the above equations, is a line may be shown by eliminating t and thereby obtaining a function of x and y, as follows: We substitute in the second equation the value $t = \frac{x - d}{b}$ obtained from the first equation. This gives, when cleared of fractions,

$$ax + by + c = 0 \qquad (2)$$

where $c = -ad - bk$.

The slope of the line defined by equation (2) is $-a/b$. The same result may also be obtained from equations (1) by recalling a property of the derivative of a function of a function;* *viz.*, if y is a function of t and t is a function of x, $D_xy = D_ty \cdot D_xt$. This may also be correctly written (for values of t for which $D_tx \neq 0$),

$$D_xy = \frac{D_ty}{D_tx}.$$

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* Vol. I, p. 134.

In the case in question, we get

$$D_t y = -a,\ D_t x = b,\ \therefore D_x y = -\frac{a}{b}.$$

A third example is given by the parametric equations

$$x = 2t,$$
$$y = \tfrac{1}{4}t^2 - 4.$$

For this function $D_x y = \frac{1}{2}t$, a linear function of t and consequently a linear function of x, since $x = 2t$. Its graph is then a parabola with a vertical axis. Elimination of t gives the Cartesian equation $y = \frac{1}{16}x^2 - 4$.

More generally, when we have two equations which express x as a linear function of t and y as a quadratic function of t, *viz.*

$$x = mt + n,$$
$$y = rt^2 + st + v,$$

the equation obtained by eliminating t is of the type

$$y = ax^2 + bx + c.$$

On the other hand, if each of x and y is given as a quadratic function of x; *e.g.*,

$$x = at^2 + bt + c,$$
$$y = dt^2 + et + f,$$

the elimination of t gives a general equation of the second degree whose graph is a conic.

The student will be able to carry through the discussion to cases where x and y are expressed as polynomials in t of degrees m and n.

Similarly, if x and y are rational functions of t, the function y of x thus defined will in general have poles. A simple example is the following:

$$x = \frac{2}{t},$$
$$y = \frac{1}{t^2 - 1}.$$

When t is eliminated, the result is

$$y = \frac{x^2}{4 - x^2},$$

a rational function with a double zero at $x = 0$ and with simple poles at $x = \pm 2$.

The parametric representation of algebraic curves is relatively unimportant since it is always possible to eliminate the parameter by rational steps, thus obtaining an algebraic equation $f(x, y) = 0$ which shows at once the relation of x and y. When the functions to be studied are **transcendental functions,** *i.e.*, non-algebraic, it frequently happens that the parametric representation is simpler than the Cartesian. Examples will now be given.

67. Parametric equations. Certain special plane curves. The cycloid. Let a circle on which a point P has been marked roll on a straight line. The point P moves on a curve which is called a cycloid. Its coordinates will be obtained in terms of the angle θ through which the radius $CP = a$ turns as the circle rolls on the line OX (Fig. 67) starting at O.

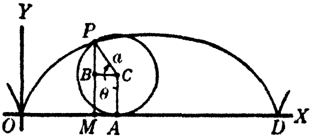

FIG. 67.

Draw $MP = y$ and BC parallel to OX. The angle $ACP = \theta$ is the angle through which the circle has turned since P was coincident with O. Hence the arc AP, whose central angle is θ, is equal in length to the line segment OA; *i.e.*, $OA = a\theta$. Then

$$x = OM = OA - MA = OA - BC = a\theta - a\sin\theta = a(\theta - \sin\theta),$$

$$y = MP = MB + BP = AC + BP = a - a\cos\theta = a(1 - \cos\theta).$$

Note. Angle $BCP = \theta - 90°$; hence $\sin BCP = -\cos\theta$ and $\cos BCP = \sin\theta$.

The student can show that the signs are correct for positions of P which correspond to values of θ less than 90° and for positions of P which correspond to values of θ greater than 180°. In all cases the coordinates of P are given by the two equations

$$x = a(\theta - \sin\theta),\quad y = a(1 - \cos\theta). \tag{3}$$

The hypocycloid. A circle of radius a rolls on the inside of a circle of radius b. The locus of a point on the rolling circle is called a hypocycloid. Its equations are obtained as follows:

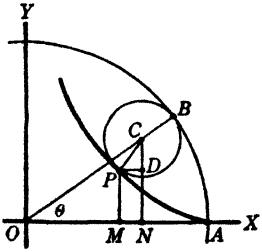

FIG. 68.

Let the circle start with P on A and roll to the position in Fig. 68. The radius OB of the fixed circle passes through the center of the rolling circle and makes an angle θ with the x-axis. The abscissa x and the ordinate y of the point P are obtained by noting that arc PB = arc AB; hence $a \cdot$ angle $PCB = b\theta$ and, consequently, angle $PCB = \frac{b\theta}{a}\cdot$ Also angle DCP + angle OCN + $\frac{b\theta}{a} = 180°$; hence angle $DCP = 180° - \frac{b\theta}{a} - (90° - \theta) = 90° - \left(\frac{b-a}{a}\right)\theta$. We have

$$x = OM = ON + NM = OC\cos\theta + a\sin DCP,$$
$$y = MP = NC - DC = OC\sin\theta + a\cos DCP;$$

i.e.,

$$\begin{aligned} x &= (b-a)\cos\theta + a\cos\frac{b-a}{a}\theta, \\ y &= (b-a)\sin\theta - a\sin\frac{b-a}{a}\theta. \end{aligned} \tag{4}$$

If $b = ka$, the curve will have k cusps and will close. In particular, when $k = 4$ we have the 4-cusped hypocycloid.

EXERCISE XLVIII

1. Make the graphs of the curves whose equations follow and find the Cartesian equation for each curve.

(*a*) $x = t,\quad y = 3t^3 - 2t - 1;$

(*b*) $x = a\sec\theta,\quad y = b\tan\theta;$

(*c*) $x = a\cos\theta,\quad y = a\sin\theta;$

(*d*) $x = a\cos\theta,\quad y = b\sin\theta;$

(*e*) $x = \frac{2}{1+t^2},\quad y = \frac{2t}{1+t^2};$

(*f*) $x = -\frac{8t}{4+t^2},\quad y = -\frac{4t^2}{4+t^2}.$

2. Obtain the Cartesian equation for the hypocycloid with four cusps. *Ans.* $x^{2/3} + y^{2/3} = b^{2/3}$.

Hint. Put $a = \frac{1}{4}b$ in equations (4), replace $\cos 3\theta$ by an expression in $\cos\theta$, and replace $\sin 3\theta$ by an expression in $\sin\theta$.

3. An **epicycloid** is the locus of a point on a circle which rolls on the outside of a fixed circle. If the radius of the rolling circle is a and that of the fixed circle is b, show that the coordinates of P are given by

$$x = (a+b)\cos\theta - a\cos\frac{a+b}{a}\theta,$$
$$y = (a+b)\sin\theta - a\sin\frac{a+b}{a}\theta.$$

The curve is closed when a and b are commensurable.

4. Derive the Cartesian equation for the cycloid.

Ans. $x = a \text{ arc sin } \dfrac{\sqrt{2ay - y^2}}{a} - \sqrt{2ay - y^2}$, or

$$x = a \text{ arc vers } y/a - \sqrt{2ay - y^2}.$$

5. Show that the cissoid of Diocles is the locus of $P(x, y)$ where

$$x = \frac{2t^2 - 2}{1 + t^2}, \quad y = \frac{2t^3 - 2t}{1 + t^2}.$$

Hint. Find the Cartesian equation of the cissoid and show that it is identically satisfied by the above expressions. What does the parameter t represent?

6. What is the locus of the point P in Fig. 69 if AP is a given length b, $BP = a$, and the line takes all possible positions such that A remains on $X'X$ and B on $Y'Y$.

Ans. The locus of P is the ellipse $x = a \cos \theta$, $y = (a + b) \sin \theta$ where θ is the angle XAB.

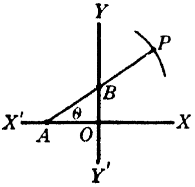

Fig. 69

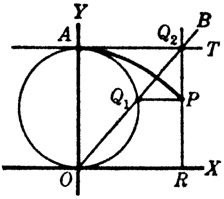

Fig. 70

7. The **witch** is the locus of the point P constructed as follows: Let a circle whose radius is a rest on the x-axis at O (Fig. 70), and let AT be the tangent to the circle at the end of the diameter OA.

Draw any line OB to meet the circle at Q_1 and the line AT at Q_2. Draw Q_2R perpendicular to OX and draw Q_1P parallel to OX. The locus of P is the so-called witch of Agnesi. Show that its equation is $y(x^2 + 4a^2) = 8a^3$ and sketch the graph.

8. Find the locus of a point whose ordinate is k times the ordinate of a point on the circle $x^2 + y^2 = a^2$. Discuss the cases $k < 1$ and $k > 1$.

9. Lines are drawn from the point $(3, 0)$ to the hyperbola $x^2 - 9y^2 = 36$. Find the locus of the mid points of these lines.

10. Show that the locus of the intersections of the perpendicular tangents to a parabola is the directrix of the parabola.

11. Show that the point of intersection of a tangent to an ellipse with a line drawn through a focus perpendicular to the tangent lies on the major auxiliary circle of the ellipse.

12. Show that the statement in Problem 11 is true also for the hyperbola.

13. Show that the point of intersection of a tangent to an equilateral hyperbola and a line drawn through the center perpendicular to that tangent lies on the lemniscate whose equation is $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

14. From the origin draw a line perpendicular to a tangent to the parabola $y^2 = -4a(x + a)$. Show that the foot of the perpendicular lies on the **strophoid**. The equation of the strophoid is $y^2 = x^2(a + x)/(a - x)$. Sketch the curve.

15. Discuss the graph of the equation $x^{1/2} + y^{1/2} = \pm a^{1/2}$.

Hint. Rotate the axes through 45°.

68. Pencils of curves. Let $f(x, y) = 0$ and $F(x, y) = 0$ be the equations of two curves in the xy-plane. The coordinates of a point $P_1(x_1, y_1)$ which is an intersection of the two curves satisfy both equations; *i.e.*, $f(x_1, y_1) = 0$ and $F(x_1, y_1) = 0$. Consequently, the coordinates of P_1 satisfy the equation

$$f(x, y) + kF(x, y) = 0. \tag{5}$$

For every value of k equation (5) *is satisfied by the coordinates of each point common to the two curves* $f(x, y) = 0$ *and* $F(x, y) = 0$.

The parameter k can take an infinite number of real values; hence equation (5) represents an infinite number of plane curves. Such a system of curves is called a **pencil,** or a **one-parameter family of curves.** We shall study the properties of some pencils of curves.

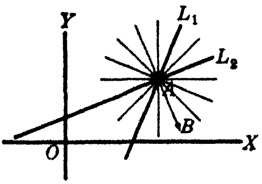

FIG. 71

Pencils of lines. Let L_1 and L_2 be two intersecting lines (Fig. 71) whose equations are, respectively:

$$\begin{aligned} L_1: &\quad a_1x + b_1y + c_1 = 0, \\ L_2: &\quad a_2x + b_2y + c_2 = 0. \end{aligned}$$

Let A be the point common to the two lines. Then, for every value of k, the linear equation

$$a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0 \tag{6}$$

represents a line through A. Consequently, the pencil of lines represented by equation (6) consists of all of the lines through A. The point A is called the **vertex of the pencil.**

Let B be any point in the plane not on L_1 or L_2. The line determined by A and B is one of the pencil of lines represented by equation (6). Its equation can be obtained as follows: Substitute the coordinates of B in (6), obtaining a linear equation in k. Since the line whose equation we seek is to pass through B, the result of substituting the coordinates of B in the left member of (6) must be zero. Thus a value of k, say k_1, is obtained which makes the left member of (6) vanish. The equation

$$a_1x + b_1y + c_1 + k_1(a_2x + b_2y + c_2) = 0,$$

in which the parameter k is replaced by the particular value k_1, gives the one of all the lines of the pencil which passes through B.

The above discussion shows that the following theorem is true:

Theorem. *Through an arbitrary point in the plane passes one and only one line of a pencil of lines.*

The only exception to the theorem is the vertex of the pencil itself.

It is customary to use symbols for the two lines, as was done in equation (5). The pencil in Fig. 71 may be represented by

$$L_1 + kL_2 = 0 \tag{7}$$

where it is understood that L_1 is merely an **abridged notation** which is used instead of the left member $a_1x + b_1y + c_1$ of the equation of which L_1 is the graph.

Let L_3 be the symbol used to represent the line of the pencil in Fig. 71 which passes through B. Obviously the pencil $L_1 + mL_3 = 0$ consists of all of the lines through the intersection of L_1 and L_3; *i.e.*, it is identical with the pencil (7). In fact the totality of the lines of the pencil with vertex at A is equally well expressed by $L' + tL'' = 0$ where L' and L'' are any two lines of the pencil.

The student will note that it is not necessary to know the coordinates of A in order to write the equation of the pencil of lines whose vertex is A.*

ILLUSTRATIVE EXAMPLE 1 Write the equation of the pencil of lines through the intersection of $2x - y + 4 = 0$ and $3x + 2y - 1 = 0$. Find the line of the pencil which passes through the point $(-1, 3)$.

SOLUTION The equation of the pencil is $2x - y + 4 + k(3x + 2y - 1) = 0$ The line through $(-1, 3)$ will correspond to the value of k obtained by substituting $(-1, 3)$ in the last equation; it is $k = \frac{1}{2}$. The required line has for its equation $2x - y + 4 + \frac{1}{2}(3x + 2y - 1) = 0$. This when simplified is $x + 1 = 0$.

ILLUSTRATIVE EXAMPLE 2. Write the equation of the pencil of lines through the intersection of the x-axis and the y-axis

SOLUTION. Since the equation of the x-axis is $y = 0$ and the equation of the y-axis is $x = 0$, the pencil through the origin is given by $y + kx = 0$

69. Pencils of circles. Let S_1 and S_2 be circles (Fig. 72) whose respective equations are as follows:

$$S_1: \qquad x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$
$$S_2: \qquad x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0.$$

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* Obviously if the vertex $A(a, b)$ is given, the equation of the pencil of lines with this vertex is $y - b = m(x - a)$.

For every value of k except $k = -1$, a circle is the graph of the equation

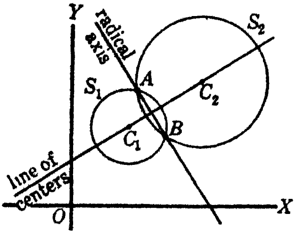

FIG. 72.

$$S_1 + kS_2 = 0, \tag{8}$$

which is an abridged form used to represent the equation

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 + k(x^2 + y^2 + 2g_2x + 2f_2y + c_2) = 0.$$

Moreover, every circle of the pencil of circles given by equation (8) passes through the two points A and B which are common to S_1 and S_2.

For $k = -1$, the x^2 terms in equation (8) cancel, as do the y^2 terms. This leaves the linear equation

$$2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0. \tag{9}$$

The line which is the graph of this equation is the line AB, since equation (9) is one of the equations obtained from (8) by giving k a special value and since each of the curves represented by equation (8) passes through A and B.

The line AB is called the **radical axis** of the two circles S_1 and S_2; it is also called the radical axis of the pencil of circles determined by S_1 and S_2.

The equation of the radical axis of the pencil of circles determined by the two circles S_1 and S_2 is $S_1 - S_2 = 0$.

The slope of the radical axis is $-(g_1 - g_2)/(f_1 - f_2)$. Let this be compared with the slope of the line of centers. The coordinates of the two centers are, respectively, $C_1(-f_1, -g_1)$ and $C_2(-f_2, -g_2)$; hence the slope of the line segment C_1C_2 is $(f_1 - f_2)/(g_1 - g_2)$. Thus *the radical axis of two circles is perpendicular to the line of centers.*

Let S_3 be a circle of the pencil (8) determined by giving to k a numerical value. Then S_3 passes through A and B and the line AB is by the last paragraph perpendicular to the line of centers C_1C_3. Hence C_3 lies on the line C_1C_2. More generally, *the centers of all of the circles of the pencil determined by two given circles S_1 and S_2 lie on the line of centers of S_1 and S_2.*

Consider two circles S_1 and S_2 which do not have any real intersections (Fig. 73). The equation

$$S_1 + kS_2 = 0$$

is a pencil of circles each of which passes through the imaginary points of intersection of S_1 and S_2. The line

$$S_1 - S_2 = 0$$

is again called the radical axis of the pencil and, as before, its slope is seen to be the negative reciprocal of the slope of the line of centers.

We shall prove an important property of the radical axis; *viz.*, *the radical axis of a pencil of circles is the locus of points from which equal tangents can be drawn to the circles of the pencil.* Before this proof can be made, however, a means of computing the length of the tangent to a circle from a given point must be found.

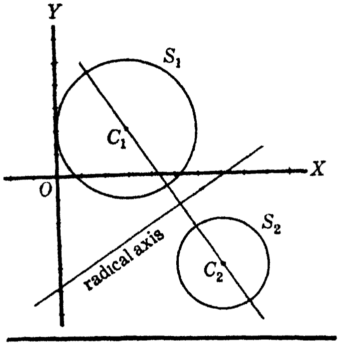

FIG. 73.

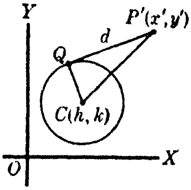

FIG. 74.

Let the circle in Fig. 74 be the graph of the equation

$$(x - h)^2 + (y - k)^2 - r^2 = 0.$$

and let the length of the tangent $P'Q$ be denoted by d. The length of CP' is $\sqrt{(x' - h)^2 + (y' - k)^2}$. In the right triangle CQP' we have

$$\mathbf{d^2 = (x' - h)^2 + (y' - k)^2 - r^2.} \tag{10}$$

The square of the length of the tangent to the circle whose equation is $(x - h)^2 + (y - k)^2 - r^2 = 0$ *or* $x^2 + y^2 + 2gx + 2fy + c = 0$ *from a point outside of the circle is obtained by substituting the coordinates of the point in the left member of the equation.*

We can now obtain the solution of the following problem: To find the locus of the point P' which has the property that the length d_1 of the tangent from P' to S_1 is equal to the length d_2 of the tangent from P' to S_2. The squares of the two lengths have the respective values

$$d_1^2 = x'^2 + y'^2 + 2g_1x' + 2f_1y' + c_1,$$
$$d_2^2 = x'^2 + y'^2 + 2g_2x' + 2f_2y' + c_2.$$

When we impose on these values the condition that they shall be equal, we have

$$2(g_1 - g_2)x' + 2(f_1 - f_2)y' + c_1 - c_2 = 0.$$

But since P' was used to indicate a general point, we may erase the prime marks and have unprimed x and y as the coordinates of the point under discussion. The equation of the locus of P is seen to be the equation $S_1 - S_2 = 0$. *q.e.d.*

EXERCISE XLIX

1. Write the equation of the pencil of lines through the point $(4, -1)$ Find the line of this pencil which passes through $(0, 2)$

2. Write the equation of the pencil of lines through the intersection of the lines $2x + y + 4 = 0$ and $x - 2y + 7 = 0$

3. Write the equation of the pencil of lines determined by $x - y + 4 = 0$ and $3x + 2y - 3 = 0$

4. Which line of the pencil in Problem 3 is also a line of the pencil in Problem 2?

5. Given two circles with centers at $(0, 3)$ and $(-4, -1)$ and respective radii equal to 4 and 5·

(*a*) Write the equation of the pencil of circles determined by them

(*b*) Write the equations of the line of centers and the radical axis

(*c*) Compute the lengths of the tangents to the two circles from the point $(6, -1)$.

6. Consider two circles with centers at $(0, 3)$ and $(-4, -1)$ and respective radii equal to 2 and 1

(*a*) Write the equation of the pencil of circles determined by them

(*b*) Find the equation of the circle of this pencil which passes through the origin.

(*c*) Write the equations of the line of centers and the radical axis

(*d*) Compute the lengths of the tangents to the two circles from the point on the radical axis whose abscissa is 6

70. Pencils of conics. Let S_1 and S_2 be two conics whose respective equations are given by

$$S_1.\quad a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0,$$
$$S_2.\quad a_2x^2 + 2h_2xy + b_2y^2 + 2g_2x + 2f_2y + c_2 = 0.$$

The equation $S_1 + kS_2 = 0$ represents a pencil of conics, each one passing through the four points in which S_1 and S_2 intersect. (The four intersections will be all real, two real and two imaginary, or all imaginary if the coefficients in the given equations are real numbers.) As before, through any point not on both of S_1 and S_2 passes a single conic of the pencil.

A value of k which makes the discriminant of the conic $S_1 + kS_2 = 0$ equal to zero corresponds to a degenerate conic in the pencil. The number of degenerate conics is the same as the number of values of k which make the discriminant vanish. This number can be found by writing down the discriminant of the pencil (*cf.* Sec. 52), *viz.*,

$$D = \begin{vmatrix} a_1 + ka_2 & h_1 + kh_2 & g_1 + kg_2 \\ h_1 + kh_2 & b_1 + kb_2 & f_1 + kf_2 \\ g_1 + kg_2 & f_1 + kf_2 & c_1 + kc_2 \end{vmatrix}.$$

When D is zero the conic degenerates. Since D is an equation of the third degree in k, there will be in general three values of k which correspond to three degenerate conics of the pencil. These degenerate conics are pairs of lines which are not necessarily real; each pair passes through the four points of intersection of S_1 and S_2.

Let these four points be designated by A, B, C, D. Then if A, B, C, and D are all distinct, the three degenerate conics consist of the following pairs of lines: AB and CD, AC and BD; AD and BC. If A, B, C, and D are not all distinct, one or more of the conics may be a pair of coincident lines.

71. The determination of circles and of conics from given conditions. The general equation of a circle, $x^2 + y^2 + 2gx + 2fy + c = 0$, contains three arbitrary constants, g, f, and c. Consequently, a circle is determined by three conditions. For example, to require that a certain point P_1 be on a circle imposes the condition $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$, an equation which is linear in the three unknowns g, f, and c. Three such equations, if consistent, determine a unique set of values for g, f, and c. Hence the equation of a circle through three given points (not all on a line) is found by solving three linear equations similar to the one above. The solution may be made by determinants.

ILLUSTRATIVE EXAMPLE. Find the equation of the circle which passes through the three points (1, 1), (−2, 3), and (−4, 0).

SOLUTION. The three conditions obtained by substituting the coordinates of the given points in turn in the general equation are

$$\begin{array}{ll} (1, 1)\cdot & 1 + 1 + 2g + 2f + c = 0, \\ (-2, 3). & 4 + 9 - 4g + 6f + c = 0, \\ (-4, 0)\cdot & 16 \quad - 8g \qquad + c = 0. \end{array}$$

The solution of these equations is $g = \frac{3}{2}, f = -\frac{1}{2}, c = -4$. Hence the equation of the required circle is

$$x^2 + y^2 + 3x - y - 4 = 0.$$

In the general equation of a conic there are six constants, a, b, c, f, g, and h. However, we can divide each term by any one of these constants which is not zero without altering the graph. Hence, we say that there are essentially five constants in the general equation of the second degree. Consequently, it takes five conditions to determine a conic; *e.g.*, we may choose five points at random and find a conic which passes through them. If the five points are real and no three are on a line, the conic is a proper conic. If three of the given points lie on a line, the conic consists of this line and a second line through the remaining two points.

When five distinct real points are given there is a very simple method of determining the conic which passes through them. An example will illustrate.

ILLUSTRATIVE EXAMPLE Find the equation of the conic which passes through the following points: $P_1(0, 1)$, $P_2(1, 4)$, $P_3(5, 0)$, $P_4(-2, -4)$, $P_5(0, 0)$.

SOLUTION. One conic through P_1, P_2, P_3, and P_4 consists of the two lines P_1P_2 and P_3P_4; another conic through the same points consists of the two lines P_1P_3 and P_2P_4. The equations of these two conics are as follows:

$$S_1: \quad (3x - y + 1)(4x - 7y - 20) = 0,$$
$$S_2: \quad (x + 5y - 5)(8x - 3y + 4) = 0.$$

The pencil of conics through these four points is given by $S_1 + kS_2 = 0$. We now impose the condition that our conic shall pass through $P_5(0, 0)$ by substituting the coefficients of P_5 in the equation $S_1 + kS_2 = 0$ This gives $k = -1$. Hence, the desired equation is $(3x - y + 1)(4x - 7y - 20) - (x + 5y - 5)(8x - 3y + 4) = 4x^2 - 62xy - 20x + 22y^2 - 22y = 0$.

EXERCISE L

1. Set up the discriminant of the pencil of conics determined by

$$S_1. \quad 4x^2 + 4xy - y^2 - 4y = 0,$$
$$S_2: \quad 4x^2 + 4xy + 5y^2 - 24x - 4y = 0.$$

Compute the three values of k for which the discriminant is zero. Find the three degenerate conics of the pencil.

Ans. $k = \frac{1}{2}, \frac{1}{4}, \frac{1}{5}$.

2. Find the equation of the conic which passes through the following points: (0, 0), (1, 4), (−3, 2), (−2, 6), (4, 0).

CHAPTER XII

PROGRESSIONS

In this chapter will be studied a new type of function, a function which is expressed as the sum of a finite number or an infinite number of terms. In the first case the function is defined only for integral values of n, where n is the number of terms.

72. Sequences; series. A finite or an infinite set of numbers or symbols, arranged in a definite order is called a **sequence.** Each symbol is called a **term** of the sequence. The totality of positive integers forms an infinite sequence; the even numbers from 20 to 40 form a finite sequence. Other examples of sequences are

$$1, 3, 5, 7, \ldots, 99.$$
$$2, 6, 18, 54, \ldots, 4374.$$
$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots$$
$$a_1, a_2, a_3, a_4, \ldots$$
$$x, x+3, x+6, x+9, \cdots$$

Two simple types of sequences are called **arithmetic progressions** and **geometric progressions.** They are defined as follows:

DEFINITIONS.

In an arithmetic progression any term after the first is formed from the preceding term by adding to it a given number called the **common difference.**

In a geometric progression any term after the first is formed from the preceding term by multiplying it by a given number called the **common ratio.**

Let the symbols a_1, a_2, a_3, . . . denote the first, second, third, . . . terms; let d denote the common difference in the arithmetic progression (A.P.); and let r denote the common ratio in the geometric progression (G.P.). Then the terms of the two progressions have the values exhibited in the following table:

	a_1	a_2	a_3	a_4		a_n	
A.P.	a_1	$a_1 + d$	$a_1 + 2d$	$a_1 + 3d$		$a_1 + (n - 1)d$	.
G.P	a_1	a_1r	a_1r^2	a_1r^3		a_1r^{n-1}	.

Hence, if a and d are known in an A.P. or if a and r are known in a G.P., the nth term can be found by

A.P. $$a_n = a_1 + (n - 1)d, \tag{1}$$

G.P. $$a_n = a_1r^{n-1}. \tag{2}$$

Each of these equations contains four symbols. The fourth can be computed if any three are known.

EXERCISE LI

1. Solve equation (1) for a_1; for d; for n

2. Solve equation (2) for a_1; for r.

3. Show that when equation (2) is solved for n, the result is

$$n = \frac{\log a_n - \log a_1 + \log r}{\log r}$$

4. In the following, values are given to three of the symbols in a progression; find the value x of the fourth symbol

	a_1	d	n	a_n
A P	-10	3	7	x
	-10	x	8	40
	x	-2	10	27
	5	3	x	41

	a_1	r	n	a_n
G P	1	2	6	x
	-3	5	x	-375
	8	x	9	$1/32$
	x	$-1/3$	8	$-1/81$

In an arithmetic progression, the terms which lie between any two terms are called the arithmetic means of the terms. Thus in the arithmetic progression $-7, -4, -1, \cdots$ the arithmetic means between -7 and 5 are -4, -1, and 2. In particular, if there is only one arithmetic mean between two terms it is called **the arithmetic mean.** In the last example, the arithmetic mean between -4 and 2 is -1. More generally, three consecutive terms of an arithmetic progression may be written

$$a_k - d, a_k, a_k + d,$$

where the arithmetic mean of $a_k - d$ and $a_k + d$ is a_k. It is half the sum of the two terms $a_k - d$ and $a_k + d$. Thus if a, b, c are three terms in an arithmetic progression, we have $b = \frac{1}{2}(a + c)$, a formula for determining the arithmetic mean.

Similarly, in the geometric progression 1, 3, 9, 27, 81, . . . the geometric mean of 1 and 9 is 3; the two geometric means between 3 and 81 are 9 and 27; etc. The formula for **the geometric mean** is derived as follows: Let a, b, c be in geometric progression. Then $c/b = b/a$, and consequently $b = \pm\sqrt{ac}$, a formula which shows that there are two numbers each of which is a geometric mean between a and b.

The arithmetic mean of two numbers is half their sum; the geometric mean of two numbers is the square root of their product.

EXERCISE LII

1. Compute the geometric mean and the arithmetic mean of 6 and 18.

2. Insert three arithmetic means between 6 and 22.

3. Insert four geometric means between 16 and $\frac{1}{2}$.

4. Insert seven arithmetic means between p and q

Ans. $\frac{7p + q}{8}, \frac{3p + q}{4}, \frac{5p + 3q}{8}, \frac{p + q}{2}, \frac{3p + 5q}{8}, \frac{p + 3q}{4}, \frac{p + 7q}{8}$.

5. In the A P 8, 5, 2, which term is -28?

6. What is the common difference of an A P. in which the first term is -32 and the fifteenth term is -4?

7. The sixth term of an A P is -9 and the tenth term is 11; find the common difference and the first term

8. Insert two different sets of three geometric means between 3 and 243.

9. The fifth term of a G P is 5 and the ninth term is $\frac{5}{8}$, find the second term.

10. In a certain school system a teacher receives $1,800 the first year and each subsequent year for five years receives an increase of $50 per year; thereafter, each year the increase is $100 per year until the maximum of $3,250 is reached How many years does the teacher work before receiving the maximum salary? What is his salary the eleventh year?

73. Sum of *n* terms. Simple formulas can be obtained for the sum S_n of n terms of a progression where n is any given positive integer. The sum of the terms of a sequence is called a **series.** For the A.P. we may write S_n in the two following ways:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n - 1)d],$$
$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + [a_n - (n - 1)d].$$

The sum of these two equations when divided by 2 is

$$S_n = n\frac{(a_1 + a_n)}{2}. \qquad (3)$$

The sum of n terms of an arithmetic progression is equal to n times the average term.

For the G.P. we may write

$$S_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1}.$$

Multiply both members of the last equation by r and get

$$rS_n = a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^n.$$

When the last equation is subtracted from the last but one the result is

$$S_n - rS_n = a_1 - a_1r^n,$$

or

$$S_n(1 - r) = a_1(1 - r^n);$$

therefore

$$S_n = \frac{a_1(1 - r^n)}{1 - r}. \tag{4}$$

74. Sum of an infinite number of terms. In an arithmetic progression it would be absurd to talk about the sum of an infinite number of terms, since the sum can be made larger than any assigned number by adding together a sufficient number of terms.

In the case of a geometric progression the situation is different. It is true that the sum grows large without bound for r greater than unity or equal to unity, but if r is numerically less than unity it can be shown that this is no longer true. Consider the difference between S_n and $\frac{a_1}{1 - r}$ as obtained from equation (4). It is

$$\frac{a_1}{1 - r} - S_n = \frac{a_1}{1 - r} \cdot r^n. \tag{5}$$

When n becomes very large, the number r^n becomes very small, as does the product of r^n by $\frac{a_1}{1 - r}$. In fact, the right member of equation (5) can be made *numerically* smaller than any previously assigned small positive number by taking n sufficiently large. This means that the sum of n terms cannot exceed $\frac{a_1}{1 - r}$ if a_1 is positive but that by taking n sufficiently large the sum of n terms may be made to differ from $\frac{a_1}{1 - r}$ by *as little as we please.*

The number $\frac{a_1}{1 - r}$ *is called the limit of the sum of n terms as n approaches infinity,*

$$\lim_{n \to \infty} S_n = \frac{a_1}{1 - r}, \qquad (|r| < 1) \quad (6)$$

or, for brevity, *the sum of an infinite number of terms.*

The student should bear in mind that when it is said that the sum of an infinite number of terms is $\frac{a_1}{1 - r}$ it is to be understood to mean that the number $\frac{a_1}{1 - r}$ is the *limit* of the sum of n terms as n increases indefinitely. He should also remember that formula (6) holds only for values of r which are numerically less than unity.

EXERCISE LIII

1. Find S_n when (1) $a_1 = 2$, $n = 8$, $d = -3$; (2) $a_1 = -2$, $n = 7$, $d = 5$; (3) $a_1 = 8$, $r = 3$, $n = 4$; (4) $a_1 = 4$, $r = 1/4$, $n = 5$.

2. On Jan. 1, a man deposited a dollar in a savings bank; on Feb. 1, he deposited two dollars; on Mar. 1, four dollars; etc. What was the sum of his deposits for a year, assuming that he continued in this manner?

3. A student borrows \$1,000 at the beginning of his senior year. He agrees to pay \$100 of the principal at the end of each year, as long as necessary. Also he agrees to pay, at the end of each year, 4 per cent interest on the principal outstanding during that year. What is the total sum which he pays?

4. Find the sum of all odd integers from 1 to 99, inclusive.

5. Find the sum of the first n odd integers.

6. Find the sum of the first n even integers.

7. At a bazaar, tickets numbered from 1 to 500 were sold; the price paid for each ticket was the number of cents equal to the number on the ticket. How much money was received for the 500 tickets?

8. A father's will directed that each of his eight children be given \$1,000 more than the child next in age but older. The fifth child received \$8,760; what was the total amount received by the group of eight?

9. Find five numbers forming an A.P. such that $S_5 = 40$ and $a_5 - a_1 = 12$.

10. Find the limit of the sum of each of the following progressions with infinitely many terms:

(*a*) $8, 2, \frac{1}{2}, \ldots;$ (*b*) $16, -4, 1, \cdots;$

(*c*) $2, \frac{2}{5}, \frac{2}{25}, \ldots;$ (*d*) $0.3, 0.03, 0.003, \ldots$

11. A pendulum is being brought to rest by air resistance; the path of each swing, after the first, of the bob of the pendulum is 0.01 shorter than that of the previous swing. If the path of the first swing is 18 in. long, find the total distance traveled by the bob in coming to rest.

12. If $a, b, c, \ldots$ are terms in a G.P., show that $\log a, \log b, \log c, \ldots$ are terms in an A.P.

13. An **unending repeating decimal** may be written as the sum of an infinite number of terms of a geometric progression where r is less than unity. For example, $0.125125125 \cdots$ is equal to the G.P. $0.125 + 0.000125 + 0.000000125 + \cdots$ The limit of the sum of the G.P. is

$\frac{0.125}{1 - 0.001} = \frac{125}{999}$; hence the number 0.125125 · · · may be written as the rational fraction $^{125}/_{999}$. *Every repeating decimal may be written as a rational fraction;* certain rational fractions are equal to terminating decimals, as, $^{1}/_{8} = 0.125$, other rational fractions, as $^{125}/_{999}$ in the example, when reduced to decimals never terminate.

Find the rational fractions which are equal to the following non-terminating repeating decimals:

(a) 1.666 . . . ;
(b) 0.121212 . . . ;
(c) 1.0101 ;
(d) 0.008333 . . . ;
(e) 2.222 . . . ;
(f) 0.636363 . . . ;
(g) 0.818181 ;
(h) 0.58333 . . . ;
(i) 0.090909 . . ;
(j) 0.429571429571 . . .

75. Harmonic progressions. A sequence of terms forms a progression called **harmonic progression** if the reciprocals of the terms form an arithmetic progression. For example, the arithmetic progression whose terms are

$$-6, \quad -2, \quad 2, \quad 6, \quad 10, \cdots$$

gives rise to the harmonic progression whose terms are

$$-\tfrac{1}{6}, \quad -\tfrac{1}{2}, \quad \tfrac{1}{2}, \quad \tfrac{1}{6}, \quad \tfrac{1}{10}, \cdots$$

In order to find the nth term of a harmonic progression it is necessary first to find the nth term of the corresponding arithmetic progression.

The name harmonic progression is due to the fact that, if the lengths of a set of strings of the same diameter and substance stretched to uniform tension form a harmonic progression, a harmonious sound results if two or more strings are caused to vibrate simultaneously.

ILLUSTRATIVE EXAMPLE. Insert four harmonic means between $-^{3}/_{5}$ and $^{3}/_{10}$.

SOLUTION. In the A.P. where $a_1 = -^{5}/_{3}$, $a_6 = ^{10}/_{3}$, and $n = 6$ we have $d = \frac{a_n - a_1}{n - 1} = \frac{1}{5}\left[\frac{10}{3} + \frac{5}{3}\right] = 1$. Hence the corresponding A.P. is $-^{5}/_{3}$, $-^{2}/_{3}$, $^{1}/_{3}$, $^{4}/_{3}$, $^{7}/_{3}$, $^{10}/_{3}$ and the required H.P. is $-^{3}/_{5}$, $-^{3}/_{2}$, 3, $^{3}/_{4}$, $^{3}/_{7}$, $^{3}/_{10}$.

EXERCISE LIV

1. If a, b, c form a harmonic progression prove that $b = \frac{2ac}{a + c}$.

2. Insert three harmonic means between $-^{1}/_{8}$ and $^{1}/_{12}$.

3. Insert six harmonic means between 2 and $^{1}/_{11}$.

4. Compute the arithmetic mean, the geometric mean, and the harmonic mean of 3 and 12; of r and s. Show that the geometric mean is a mean proportional between the arithmetic and harmonic means.

5. Let m, n, r, and s form a harmonic progression. Prove that $mn \div rs = (m - n) \div (r - s)$.

SUMMARY

- Definitions of new terms.
  - Sequence.
  - Progression.
    - Arithmetic.
    - Geometric.
    - Harmonic.
- Formulas.
  - The *n*th term.
  - The sum of n terms.
  - The sum of an infinite number of terms.
  - The arithmetic mean of a and b.
  - The geometric mean of a and b.
  - The harmonic mean of a and b.

CHAPTER XIII

MATHEMATICAL INDUCTION. THE BINOMIAL THEOREM

When a general formula is to be proved it is frequently necessary to use a method called the method of mathematical induction. This method will be described now and will be applied to various types of problems; in particular, the formula, called the binomial formula, which gives the expansion of the expression $(a + b)^n$ as a polynomial in a and b when n is a positive integer, will be obtained.

76. Mathematical induction. *The sum of the first n integers is $n/2(n + 1)$*; this statement will be proved by mathematical induction.

To prove.

$$1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n + 1). \tag{1}$$

There are three steps in the proof, as follows:

I. *The formula is shown to be true for some convenient small value of n, say $n = 2$.*

II. *The formula is assumed to be true for some given value of n, say $n = k$.*

III. *On the basis of the assumption in II it is proved that the formula holds for $n = k + 1$.*

Conclusion. By I, the formula is true for $n = 2$; hence by III it is true for $n = 3$; but III asserts that since the formula is true for $n = 3$ it is also true for $n = 4$; furthermore, by III since the formula is true for $n = 4$, it is also true for $n = 5$; *etc.* Thus by advancing one unit at a time, we can show that the formula is true for any previously chosen positive integral value of n; *i.e.*, it is true for any positive integer n.

Let us apply the three steps of the proof to the example given in equation (1).

I. When $n = 2$ we have $1 + 2 = \frac{1}{2} \cdot 2(2 + 1)$; the formula is true for $n = 2$.

II. We assume $1 + 2 + 3 + \cdots + k = \frac{1}{2}k(k + 1)$. (2)

III. We wish to prove $1 + 2 + 3 + \cdots + (k + 1) = \frac{1}{2}(k + 1)(k + 2)$.

Proof. Let $k + 1$ be added to each member of equation (2). This does not disturb the equality and it gives

$$1 + 2 + 3 + \cdots + k + (k + 1) = \tfrac{1}{2}k(k + 1) + (k + 1),$$
$$1 + 2 + 3 + \cdots + k + (k + 1) = \tfrac{1}{2}(k + 1)(k + 2).$$

Hence the conclusion follows that the formula is true for $n = 3$; for $n = 4$; for $n = 5$; $\cdots$; for $n =$ *any positive integer.* *q.e.d.*

ILLUSTRATIVE EXAMPLE 2. Prove the formula $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ by mathematical induction.

I. For $n = 2$ the formula gives $1 + 3 = 2^2$.

II. We assume $1 + 3 + 5 + \cdots + (2k - 1) = k^2$.

III. Let the next odd number, $2k + 1$, be added to both members of the last equation; the result is $1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$ which is the result obtained by setting $n = k + 1$ in the formula. Hence, if the formula is true for $n = k$, it is surely true for $n = k + 1$. But by I it is true for $n = 2$; hence by successive applications of III we conclude that the formula is true for $n = 3$, for $n = 4$, $\cdots$, for $n =$ any positive integer. *q.e.d.*

From these examples it is clear that the method of mathematical induction is used only when a formula is stated in terms of a number symbol n which is to be allowed to take the successive values 1, 2, 3, . . .

For example, the induction method is not needed in obtaining the quadratic formula, since in this formula appear the coefficients, a_0, a_1, a_2, of the general quadratic but these symbols are not supposed to take *successive* integral values. There is also a formula for the three roots of the general cubic equation in one unknown and another formula for the four roots of the general quartic equation in one unknown. For many years mathematicians sought a formula for the n nth roots of the general n-ic equation in one unknown. They hoped to find a formula which would contain the symbol n as well as the coefficients appearing in the equation. By allowing n to take the successive values 1, 2, 3, . . . it was hoped that the one formula would do for equations of all degrees. It was Abel,* an eminent Norwegian mathematician, who proved that there can be no such formula, *i.e.*, a formula involving only the operations of arithmetic.

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* Niels Henrik Abel was born in 1802 and died in 1829. In this brief period he made numerous important contributions to mathematical knowledge.

EXERCISE LV

1. Prove the following formulas:

(a) $2 + 4 + 6 + \cdots + 2n = n(n + 1)$;
(b) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$;
(c) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$;
(d) $2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2}{3}n(n + 1)(2n + 1)$;
(e) $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{1}{3}n(2n + 1)(2n - 1)$;
(f) $1^3 + 2^3 + 3^3 + \cdots + n^3 = [\frac{1}{2}n(n + 1)]^2$;
(g) $2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 + \cdots + (n + 1)(n + 4) = \frac{1}{3}n(n + 4)(n + 5)$;
(h) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}$.

2. Prove by mathematical induction the formula for the sum of n terms in an arithmetical progression; in a geometrical progression

3. Prove De Moivre's theorem. (*Cf.* Chap I, Sec 7.)

77. The binomial theorem. A general formula can be obtained for the terms in the expansion of $(a + b)^n$ where n is a positive integer. It is

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n - 1)}{1 \cdot 2}a^{n-2}b^2 + \cdots$$
$$\frac{n(n - 1) \cdots (n - k + 2)}{1 \cdot 2 \cdots (k - 1)}a^{n-k+1}b^{k-1} + \cdots + nab^{n-1} + b^n. \quad (3)$$

The proof will be made in Sec. 78 by mathematical induction. First, however, let us examine the right member of equation (3). We note the following:

a. Terms. There are $n + 1$ terms.

b. Exponents. 1. The exponent of a is reduced one unit in each succeeding term after the first term, a^n.

2. The exponent of b is zero in the first term, it is 1 in the second term. It is $k - 1$ in the kth term. $(k \leqq n + 1.)$

3. The sum of the exponents of a and b in any term is n.

c. Coefficients. 1. The coefficient of the second term is n.

2. The denominator of the coefficient of the second term is $1!$, of the third term is $2!$, of the kth term is $(k - 1)!$ $(k = 0, 1, 2, \cdots, n + 1)$.

3. There are as many factors in the numerator of a coefficient as in the denominator. The first factor is n.

From the above, it is easy to write down the terms in the expansion of a given binomial. Furthermore, any desired term can be written down without first writing the preceding terms.

ILLUSTRATIVE EXAMPLE 1. Write the fifth term of $(2x + y)^9$.

SOLUTION. The fifth term contains y^4 and consequently it contains $(2x)^5$; the denominator of the coefficient is $1 \cdot 2 \cdot 3 \cdot 4$; the numerator of the coefficient is $9 \cdot 8 \cdot 7 \cdot 6$. The term is

$$\frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4}(2x)^5(y)^4 = 4032x^5y^4.$$

ILLUSTRATIVE EXAMPLE 2. Write the eighth term of $(3a - 4b)^{12}$.

SOLUTION. The eighth term contains $(-4b)^7$ and consequently it contains $(3a)^5$. The coefficient has 7 factors in the numerator and 7! in the denominator. The term is

$$\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}(3a)^5(-4b)^7 = -49268736a^5b^7.$$

EXERCISE LVI

1. Write the following expansions·

(*a*) $(3x - 1)^5$; (*b*) $(2x + y)^7$; (*c*) $(x^2 - y^2)^{10}$.

2. Write the sixth term of the expansion of $\left(4t - \frac{1}{4t}\right)^9$; of $\left(6z^2 + \frac{1}{2}w^2\right)^{12}$.

3. Is the fifth term of the expansion of $(x - y)^{11}$ positive or negative?

4. Which terms in the expansion of $(x - y)^{10}$ are negative?

5. Compare the coefficients of the sixth and the thirteenth terms in the expansion of $(a - b)^{18}$; of $(a - b)^{19}$.

6. Write the coefficients in the expansion of $(a - b)^8$; of $(a - b)^9$.

7. Show how the binomial theorem can be used to get the expansion of $(a + b + c)^n$.

78. Proof of the binomial formula. We wish to prove formula (3) in the last section by the method of mathematical induction.

I. The theorem is true for $n = 2$, for by actual multiplication we get $(a + b)^2 = a^2 + 2ab + b^2$.

II. We assume that when $n = k$ we have the equality

$$(a + b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{1 \cdot 2}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3}a^{k-3}b^3 + \cdots + b^k. \quad (4)$$

III. We multiply the two members of equation (4) by $(a + b)$; on the left we get $(a + b)^{k+1}$; on the right we get the first row below when we multiply by a, the second row when we multiply by b, and the sum below the line when similar terms are added.

$$\begin{array}{lllllll}
a^{k+1} + k & a^kb + \frac{k(k-1)}{1 \cdot 2} & a^{k-1}b^2 + \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3} & a^{k-2}b^3 + \cdots + ab^k & & \\
 & a^kb + k & a^{k-1}b^2 + \frac{k(k-1)}{1 \cdot 2} & a^{k-2}b^3 + \cdots + kab^k & + b^{k+1} \\
\hline
a^{k+1} + (k+1) & a^kb + \frac{(k+1)k}{1 \cdot 2} & a^{k-1}b^2 + \frac{(k+1)k(k-1)}{1 \; 2 \cdot 3} & a^{k-2}b^3 + \cdots + (k+1)ab^k & + b^{k+1}
\end{array}$$

On adding the terms $\frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3}a^{k-2}b^3$ and $\frac{k(k-1)}{1 \cdot 2}a^{k-2}b^3$, for example, we get for the coefficient

$$\frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3} + \frac{k(k-1)}{1 \cdot 2} = \frac{k(k-1)}{1 \cdot 2}\left[\frac{k-2}{3} + 1\right] = \frac{k(k-1)}{1 \cdot 2} \cdot \frac{k+1}{3}.$$

The expansion of $(a + b)^{k+1}$ obtained in this way is exactly the same as that obtained by replacing n by $k + 1$ in the formula. Hence, the formula is true for $n = k + 1$ if it is true for $n = k$. But in I it was shown to be true for $n = 2$; hence it is true for $n = 3$, for $n = 4$, $\cdots$, for $n =$ any positive integer. *q.e.d.*

Note. The binomial theorem holds also for n a negative number and for n a fraction. In these cases the expansion consists of an infinite number of terms. When $|b| < |a|$ we write $(a + b)^n = a^n\left(1 + \frac{b}{a}\right)^n$ and the expansion is an infinite geometrical progression with the ratio less than unity.

ILLUSTRATIVE EXAMPLE 1. Find $\sqrt{26}$ by expanding the binomial $(25 + 1)^{1/2}$.

SOLUTION. We have $(25 + 1)^{1/2} = (25)^{1/2}(1 + 1/25)^{1/2} = 5(1 + 1/25)^{1/2}$.

$$\left(1 + \frac{1}{25}\right)^{1/2} = 1 + \frac{1}{2} \cdot \frac{1}{25} + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2!}\frac{1}{25^2} + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{3!}\frac{1}{25^3} + \cdots$$

The terms may be replaced by decimals and added, *viz.*,

$$\begin{array}{r} 1\ 000000 \\ 0\ 020000 \\ -\ 0.000200 \\ 0.000004 \\ \hline 1.019804 \end{array}$$

Hence, $\sqrt{26}$ correct to 6 decimal places is $5(1.019804) = 5.099020$.

If only three terms of the expansion had been taken, the result would have been correct to five decimal places. The fifth term in the expansion is $\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{1}{4!}\frac{1}{25^4}$; its value is -0.0000001, so that the error in taking 5.099020 for $\sqrt{26}$ is less than 0.0000005.

ILLUSTRATIVE EXAMPLE 2. Expand $\frac{1}{1-x}$ by the binomial theorem and discuss the expansion.

SOLUTION. We have

$$f(x) = (1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots \tag{5}$$

The right member of (5) is a geometrical progression and therefore has a finite limit if $|x| < 1$. Consequently we can approximate the value of $f(x)$ for a given x which is numerically less than 1 by substitution in the right member of (5); we can get the exact value of $f(x)$ for this x by substituting in the left member of (5). For example, when $x = \frac{1}{4}$, $\frac{1}{1-x} = \frac{4}{3}$. Let us see how rapidly the sum $1 + x + x^2 + x^3 + \cdots$ approaches $\frac{4}{3}$ in value. We have

Number of terms	Sum	Difference between sum and $\frac{4}{3}$
2	1.25	$0.08\frac{1}{3}$
3	1.3125	$0.0208\frac{1}{3}$
4	1.328125	$0.005208\frac{1}{3}$
5	1.33203125	$0.00120208\frac{1}{3}$
6	1.3330078125	$0.0003255208\frac{1}{3}$

After adding six terms, the sum is correct for three decimal places only. This series is said to *converge* slowly. The series in the last Illustrative Example converges rapidly. In the series under discussion let us take $x = 0.1$; then $\frac{1}{1 - 0.1} = \frac{10}{9}$. When $x = 0.1$ is substituted in the right member of equation (5) the sum of the first four terms, $1 + x + x^2 + x^3$, is equal to 1.111, a number which differs from $\frac{10}{9}$ by $0.000\frac{1}{9}$. This series converges more rapidly for $x = 0.1$ than for $x = \frac{1}{4}$.

In the same way, such expressions as

$$\frac{1}{\sqrt{1-x}},\quad \frac{1}{1-x^2},\quad \frac{1}{\sqrt[3]{1-x^2}},\quad \frac{1}{\sqrt{a+bx+cx^2}},$$
$$\frac{1}{a + bx + cx^2 + \cdots + px^n}$$

can be expanded into infinite series. The series represents the original expression for suitably chosen values of x. Such infinite series and their regions of validity are studied in more advanced mathematical texts.

EXERCISE LVII

1. Use the binomial theorem to compute $\sqrt[3]{100}$.

Hint. Write $100 = 125 - 25$ and compute the sum of four terms of the binomial expansion. Compute the fifth term in order to get an upper limit of the error.

Next, write $100 = 64 + 36$ and compute the sum of the first four terms and the value of the fifth term as before. Discuss the results.

2. How many terms of $(32 - 4)^{\frac{1}{5}}$ must be computed in order to obtain the value of $\sqrt[5]{28}$ with an error less than 1 in the fifth decimal place?

3. Find the fourth term in the expansion of $(64 - 16)^{1/4}$. Hence compute the accuracy of the result of taking three terms in this expansion as an approximation to $\sqrt[4]{48}$

4. Check the result in Problem 3 by solving the equation $x^4 = 48$ by Horner's method.

SUMMARY

The three steps in an induction proof.

The binomial theorem.
- The exponent a positive integer.
  - Number of terms.
  - Exponents.
  - Coefficients.
- The exponent a fraction or negative number.
  - Number of terms.

CHAPTER XIV

PERMUTATIONS AND COMBINATIONS

It is sometimes desirable to know the number of different arrangements or **permutations** possible for a given set of objects. For example, the symbols a, b and c are capable of six permutations, *viz.*, abc, acb, bac, bca, cab, and cba.

For other purposes it is desirable to know how many distinct groups or **combinations** of r objects can be formed out of n objects. In this case the arrangement of the objects is not important and the six permutations listed above form only one combination.

In this chapter the thoery and applications of permutations and combinations will be studied.

79. Permutations. Fundamental principle. Let us suppose that a certain act can be performed in k ways and that after this has been performed (in *any one* of the k ways) another act can be performed in m ways. *The two acts can be performed in succession, the second after the first, in km ways.* If after these two acts have been performed a third act can be performed in n ways, the three acts can be performed in succession, in the order named, in kmn ways, etc.

ILLUSTRATIVE EXAMPLE. In how many ways can 3 books be taken from a shelf containing 12 books?

SOLUTION. There are 12 choices for the first book; after 1 book has been taken from the shelf there are 11 choices for the second; there are then 10 choices for the third. In all, 3 books may be removed in $12 \times 11 \times 10 = 1{,}320$ ways.

More generally, a set of r objects may be formed from a set of n objects in $n(n-1)(n-2)\cdots(n-r+1)$ ways, since there are n choices for the first object, $n-1$ choices for the second, etc., until r objects have been chosen.

The symbol ${}_nP_r$ is used to denote the number of permutations of n objects taken r at a time; by the argument in the last para graph we have the formula

$$ {}_nP_r = n(n-1)(n-2)\cdots(n-r+1). \qquad (1)$$

This formula can be put in a more elegant form by multiplying and dividing the right member by $(n - r)!$* This gives $n!$ in the numerator, and the formula becomes

$$_nP_r = \frac{n!}{(n-r)!}. \tag{2}$$

In particular, if $r = n$ so that every arrangement is an arrangement of the entire set of n objects, the factors on the right of equation (1) are n in number, the first factor is n and the last one is 1; *i.e.*,

$$_nP_n = n!. \tag{3}$$

It may happen that some of the objects must be treated differently from the others, as in the following.

ILLUSTRATIVE EXAMPLE 1. How many different numbers of two digits each may be formed from 0, 1, 2, 3, 4?

SOLUTION. The figure in the tens place may be chosen in four ways; *i e*, it may be any one of 1, 2, 3, or 4 After one of these figures has been chosen for the tens place, the figure in the units place may be any one of the five given figures. Hence, there are $4 \times 5 = 20$ numbers with two digits each containing no figures other than 0, 1, 2, 3, 4

ILLUSTRATIVE EXAMPLE 2. How many permutations can be made with the letters of the word Tennessee?

SOLUTION. If the nine letters of the word were all different, the number of permutations would be $9!$ However, interchanging two *e*'s, for example, does not give a different permutation and as there are $4!$ arrangements of the *e*'s, $2!$ arrangements of the *n*'s and $2!$ arrangements of the *s*'s, there are in all $4! \cdot 2! \cdot 2!$ arrangements which do not give distinguishable permutations. Hence the number of distinguishable permutations is given by

$$\frac{9!}{2!\,2!\,4!} = 3780.$$

More generally, if among the n objects of which permutations are to be made there are r_1 which are alike and r_2 others which are alike but not like the r_1 objects in the first set, etc., the number of distinguishable permutations is

$$\frac{n!}{r_1!\,r_2!\cdots}. \tag{4}$$

EXERCISE LVIII

1. In how many ways can a group of 11 men line up as a football team if 4 of the men can play in the line only, and 2 others in the backfield only?

* How is the result changed if 2 of the men can play in the line only, and 4 others in the backfield only.

2. How many numbers between 100 and 1,000 have at least one digit repeated?

3. How many five-place numbers can be formed from the digits 2, 4, 6, 8, 0, if the middle place is to be occupied always by 8?

4. An employment agency has four applicants with whom to fill two different positions. In how many ways can the positions be filled?

5. How many permutations are there of the letters in the word *Olympic* taken seven at a time?

6. Four officers of an organization are to be elected from a group of 16 candidates. In how many ways can this be done?

7. After the host and hostess at a dinner party are placed, how many arrangements are possible for eight men guests? How many arrangements are possible for four men and four women guests if two men may not sit in adjacent seats?

8. How many even numbers of three digits each may be formed from the digits 1, 2, 3, 4, 5, 7, and 9?

9. If six coins are tossed together, in how many different ways can they fall?

10. In how many ways can seven persons take places in a seven-passenger car if only three of the group can drive?

11. How many distinct permutations of four letters can be made from the letters of the word *distinct*?

12. In how many different orders can the colors red, green, yellow, and black be arranged, all four colors to appear in each arrangement?

80. Combinations. It is required to find the number of combinations of n objects taken r at a time. Consider a single combination of r objects. These objects can be arranged in $r!$ permutations. Hence the number of combinations multiplied by the number of permutations in each combination gives the total number of permutations. Let the symbol ${}_nC_r$ be used to denote the number of combinations of n things taken r at a time. Then we have the following relation:

$$r!\,{}_nC_r = {}_nP_r. \tag{5}$$

When the value of ${}_nP_r$ is substituted from equation (3), we have

$${}_nC_r = \frac{n!}{r!(n-r)!}. \tag{6}$$

When r objects are chosen from a group of n objects, there remains a group of $n - r$ objects; *i.e.*, for every combination of r objects there is a combination of $n - r$ objects. Hence,

$${}_nC_r = {}_nC_{n-r}. \tag{7}$$

ILLUSTRATIVE EXAMPLE 1. Compute the number of permutations and the number of combinations of the letters a, b, c, d taken three at a time.

SOLUTION. By formula (3) we get ${}_4P_3 = \frac{4!}{1!} = 24$; by formula (6), ${}_4C_3 = \frac{4!}{3!\,1!} = 4$.

The tabulation which follows shows clearly how the permutations fall into groups of $3! = 6$, each group corresponding to one and only one combination.

Combinations	Permutations
abc	*abc, acb, bac, bca, cab, cba.*
abd	*abd, adb, bad, bda, dab, dba.*
acd	*acd, adc, cad, cda, dac, dca.*
bcd	*bcd, bdc, cbd, cdb, dbc, dcb.*

ILLUSTRATIVE EXAMPLE 2. How many different straight lines are determined by eight points in a plane, if no three points are in a line?

SOLUTION. Two points may be chosen in $8 \cdot 7 = 56$ ways. But since the choice of A and B in this order gives the same line as the choice of B followed by A, the number of lines is $^{56}/_2 = 28$.

The binomial coefficients may be expressed by use of the symbol ${}_nC_r$. A comparison of the binomial coefficients,

$$n,\ \frac{n(n-1)}{1\cdot 2},\ \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3},\ \cdots,\ \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k},\ \cdots \qquad (8)$$

with the value of ${}_nC_r$ given in equation (6), shows that the coefficients in (8) are, respectively,

$${}_nC_1,\ {}_nC_2,\ {}_nC_3,\ \cdots,\ {}_nC_k,\ \cdots$$

Hence the binomial expansion may be written in the following form:

$$(a+b)^n = a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \cdots \qquad (9)$$

There is a second symbol which is frequently used for the binomial coefficients. It is $\binom{n}{k}$ and stands for the fraction whose denominator is $k!$ and whose numerator is $n(n-1)\cdots$ (k factors). The binomial expansion, when these symbols are used, has the following form:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots \qquad (10)$$

EXERCISE LIX

1. How many different groups of 6 people each can be selected from 13 people?

2. How many different sums of money, consisting of three coins each, can be formed from a penny, a nickel, a dime, a quarter, and a half dollar?

3. How many different hands of 13 cards each can be made from a deck of 52 cards? Leave the result in factored form.

4. From a group of eight different things, in how many different ways can we choose groups of not more than four things?

5. How many triangles can be formed whose vertices lie at three of seven given points in a plane? Assume that no three of the seven points lie on a line.

6. There are four roads from *A* to *B* and five roads from *B* to *C*. In how many ways can a person make the round trip from *A* to *C* and back again, passing through *B* each trip?

7. In how many ways can we select three golf balls and three tennis balls from five golf balls and five tennis balls?

8. Two dice are tossed together. In how many ways can they fall?

9. Three dice are tossed together. In how many ways can they fall?

10. In how many ways can eight books be arranged on a shelf if two of the books must be placed together in a given order?

CHAPTER XV

PARTIAL FRACTIONS

When two or several fractions with unlike denominators are added, the result in its lowest terms is in general a fraction with a denominator which is the lowest common denominator of the denominators of the fractions whose sum it is. For example, if A and B are numbers, the sum of $\frac{A}{x-a}$ and $\frac{B}{x-b}$ has the form

$$\frac{Cx + D}{(x-a)(x-b)}.$$

The sum of the fractions $\frac{A}{(x-a)^2}$, $\frac{B}{x-a}$, and $\frac{C}{x-b}$ as well as the sum of the fractions $\frac{A}{(x-a)^2}$ and $\frac{C}{x-b}$ has the form

$$\frac{Dx^2 + Ex + F}{(x-a)^2(x-b)}.$$

In this chapter the inverse problem will be studied, *viz.*, to resolve or decompose into the sum of two or more simple fractions a fraction, *i.e.*, a rational function of one variable, whose denominator is the product of two or more real factors of types to be discussed in the next section.

81. Preliminary notions. If the real number a is a zero of a **real polynomial,*** $x - a$ is a factor of the polynomial.

If the complex number $a + bi$ is a zero of a real polynomial, the corresponding linear factor is not a *real* factor. In this chapter we wish to work only with real polynomials; hence we shall factor our polynomial denominators only as far as it is possible to factor them into the product of real polynomials.

A real polynomial which has $a + bi$ as a zero has also $a - bi$ as a zero. These two numbers are the roots of the real quadratic equation $x^2 - 2ax + a^2 + b^2 = 0$. Hence a polynomial with a pair of conjugate complex zeros has, as a factor, a real quadratic, called an **irreducible quadratic.***

---

* A real polynomial is a polynomial whose coefficients are real numbers.

A fraction is called a **simple fraction** if it has one of the following forms:

$$(a)\ \frac{A}{x-r}, \qquad (b)\ \frac{A}{(x-r)^k}, \qquad (c)\ \frac{Ax+B}{ax^2+bx+c},$$
$$(d)\ \frac{Ax+B}{(ax^2+bx+c)^k},$$

where all of the constants are real, k is a positive integer, and $ax^2 + bx + c$ is an irreducible quadratic.

A fraction is called a **proper fraction** if the degree of the numerator is less than the degree of the denominator. The four types of simple fractions are proper.

When the degree of the denominator is not greater than that of the numerator of a rational fraction, the numerator can be divided by the denominator, obtaining a polynomial quotient and a remainder whose degree is less than the degree of the divisor; *i.e.*, an improper fraction (in its lowest terms) is the sum of a polynomial and a proper fraction.

Examples.

$$(a)\ \frac{3x^2+7}{x-2} = 3x + 6 + \frac{19}{x-2}; \qquad (b)\ \frac{x^2-3x}{2x^2+1} = \frac{1}{2} - \frac{3x+\frac{1}{2}}{(2x^2+1)}.$$

More generally, let $\dfrac{f(x)}{g(x)}$ be an improper fraction. Then we may write

$$\frac{f(x)}{g(x)} = P(x) + \frac{r(x)}{g(x)}, \tag{1}$$

where each of the functional symbols in equation (1) represents a real polynomial and where the degree of $r(x)$ is less than that of $g(x)$.

Equation (1) is called an **identical equation**; the two members are equal for all values of x for which they are defined, *i.e.*, for all values of x except the zeros of $g(x)$.

An equation which is not an identical equation is called a **conditional equation**; the two members are equal for a finite number of values of x.

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* A reducible quadratic can be written as the product of two real linear factors; *e.g.*, $x^2 - 9$ is reducible, $x^2 + 9$ and $x^2 + x + 4$ are irreducible. (*Cf.* the last sentence in Sec. 11.)

Examples. The equation $x^2 - 5x - 14 = (x - 7)(x + 2)$ is an identical equation, as is $\frac{x^2 + 2}{x} = x + \frac{2}{x}$. The first identity holds for all values of x; the second for all values of x except $x = 0$, for which value neither member of the equation is defined. The equation $\frac{x^2 + 2}{x} = 5x + \frac{1}{x}$ is a conditional equation. The members are not defined for $x = 0$, they are equal only for $x = \frac{1}{2}$ and $x = -\frac{1}{2}$.

Let the two members of equation (1) be multiplied by $g(x)$. The result is

$$f(x) = P(x)g(x) + r(x), \tag{2}$$

an identity which holds for all values of x. Since in equation (2) the left member is a polynomial, when the multiplication indicated on the right is carried out and the product is added to $r(x)$, the resulting expression must be the same polynomial as the one on the left, *i.e., corresponding powers of x have equal coefficients.*

Let us see how the preceding definitions and remarks can be applied to the decomposition of proper fractions into simple partial fractions.

82. Decomposition. Only fractions whose denominators can be factored into a product of factors which are linear factors or irreducible quadratic factors with rational coefficients or which are a combination of the two will be considered. Some examples will show the method of obtaining the partial fractions.

ILLUSTRATIVE EXAMPLE 1. Resolve into its partial fractions $\frac{2x - 1}{x^2 + x - 6}$.

SOLUTION. We write $\frac{2x - 1}{x^2 + x - 6} = \frac{A}{x + 3} + \frac{B}{x - 2}$, where A and B are number symbols whose values are to be determined. Let both members be multiplied by $(x + 3)(x - 2)$; the result is the identical equation

$$2x - 1 = A(x - 2) + B(x + 3). \tag{3}$$

When a number is substituted for x, equation (3) gives a relation connecting A and B. Two such relations are sufficient to determine the values of these symbols. The values of x which have been used in order to determine the following relations are set off at the side and are kept there as a record of what has been done.

$x = 2$: $\quad 3 = 5B$, $\quad \therefore B = \frac{3}{5}$;

$x = -3$: $\quad -7 = -5A$, $\quad \therefore A = \frac{7}{5}$.

Thus the desired decomposition is

$$\frac{2x - 1}{(x + 3)(x - 2)} = \frac{7}{5(x + 3)} + \frac{3}{5(x - 2)}. \tag{4}$$

It is important and quite simple to check the above computation. The student can have confidence in the correctness of his results if, when he substitutes in equation (4) a value of x not used in determining A and B, he gets the same value for both members of the equation. For $x = 0$ the left member of (4) is $1/6$, for $x = 0$ the right member is $7/15 - 3/10 = 1/6$.

ILLUSTRATIVE EXAMPLE 2. Resolve into its partial fractions

$$\frac{4x^3 - 14x^2 + 18x - 9}{(x-2)^2(x-1)^2}.$$

SOLUTION.

$$\frac{4x^3 - 14x^2 + 18x - 9}{(x-2)^2(x-1)^2} = \frac{A}{(x-2)^2} + \frac{B}{x-2} + \frac{C}{(x-1)^2} + \frac{D}{x-1}.$$

$$4x^3 - 14x^2 + 18x - 9 = A(x-1)^2 + B(x-2)(x-1)^2 + C(x-2)^2 + D(x-1)(x-2)^2.$$

$x = 1$: $\quad -1 = C;$

$x = 2$: $\quad 3 = A;$

$x = 0$: $\quad -9 = A - 2B + 4C - 4D, \qquad -2B - 4D = -8;$

$x = -1$: $\quad -45 = 4A - 12B + 9C - 18D, \qquad -2B - 3D = -8.$

From the last two equations we get $D = 0$ and $B = 4$. Hence, the result of the decomposition is as follows:

$$\frac{4x^3 - 14x^2 + 18x - 9}{(x-2)^2(x-1)^2} = \frac{3}{(x-2)^2} + \frac{4}{x-2} - \frac{1}{(x-1)^2}.$$

Check. For $x = 2$ we get $27/4 = 3 + 4 - 1/4$ The check is satisfactory.

ILLUSTRATIVE EXAMPLE 3. Decompose into its partial fractions

$$\frac{5x^4 + 7x^2 - 3x + 1}{x(x^2+1)^2}.$$

SOLUTION. We write

$$\frac{5x^4 + 7x^2 - 3x + 1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{(x^2+1)^2} + \frac{Dx + E}{x^2+1},$$

$$5x^4 + 7x^2 - 3x + 1 = A(x^2+1)^2 + (Bx + C)x + (Dx + E)x(x^2+1).$$

$x = 0$: $\quad 1 = A, \qquad A = 1;$

Coefficients of x^4: $\quad 5 = A + D, \qquad \therefore D = 4;$

Coefficients of x^3: $\quad 0 = E, \qquad E = 0;$

Coefficients of x: $\quad -3 = C + E, \qquad \therefore C = -3;$

$x = 1$: $\quad 10 = 4A + B + C + 2D + 2E.$

Substituting the values already obtained, we get

$$B = 1.$$

Hence, we have

$$\frac{5x^4 + 7x^2 - 3x + 1}{x(x^2+1)^2} = \frac{1}{x} + \frac{x - 3}{(x^2+1)^2} + \frac{4x}{x^2+1}$$

Check. For $x = -1$ the left member becomes $\frac{16}{-4} = -4$; the right member becomes $-1 - \frac{4}{4} - \frac{4}{2} = -4$

ILLUSTRATIVE EXAMPLE 4 Decompose into simple partial fractions

$$\frac{2x - 5}{(x + 3)^4}$$

SOLUTION It is not necessary to use undetermined coefficients when there are fewer than two distinct factors in the denominator. We write

$$\frac{2x - 5}{(x + 3)^4} = \frac{2x + 6 - 11}{(x + 3)^4} = \frac{2x + 6}{(x + 3)^4} - \frac{11}{(x + 3)^4} = \frac{2}{(x + 3)^3} - \frac{11}{(x + 3)^4}$$

ILLUSTRATIVE EXAMPLE 5. Decompose into simple partial fractions

$$\frac{5x^3 + 7x^2 - 6x - 10}{(x^2 - 2x + 4)^2}$$

SOLUTION We write

$$\frac{5x^3 + 7x^2 - 6x - 10}{(x^2 - 2x + 4)^2} = \frac{5x^3 - 10x^2 + 20x + 17x^2 - 34x + 68 + 8x - 78}{(x^2 - 2x + 4)^2}$$

$$= \frac{5x + 17}{x^2 - 2x + 4} + \frac{8x - 78}{(x^2 - 2x + 4)^2}$$

EXERCISE LX

1. Use the method of undetermined coefficients to find the simple partial fractions whose sums are equal to the following proper fractions

(*a*) $\frac{2x^2 + 16x - 76}{(x + 1)(x - 2)(x - 5)}$, (*b*) $\frac{2x^4 + 2x^2 - 11x + 15}{x^3(x - 3)}$,

(*c*) $\frac{3x - 49}{x^2 - 49}$, (*d*) $\frac{7x^3 + 16x^2 + x - 13}{(x + 2)^2(x + 1)^3}$,

(*e*) $\frac{x^3 - 6x^2 + x + 2}{x^2(3x^2 - x - 1)}$; (*f*) $\frac{4x^5 + 21x^3 - 42x^2 - x - 185}{(x + 1)^2(x^2 + 5)^2}$,

(*g*) $\frac{12}{(x + 10)(x - 6)}$; (*h*) $\frac{2x^2 + 3x}{(4x + 9)(x - 3)}$;

(*i*) $\frac{x^2 + 5x - 4}{(x - 3)^2(x + 7)}$; (*j*) $\frac{7x^4 + 32x^2 + 30}{(x^2 + 1)^2(3x^2 + 4)}$

2. Resolve the following into simple partial fractions without using undetermined coefficients

(*a*) $\frac{3x^5 - 14x^4 + 2x^3 - 12x^2 - 3}{(x - 5)^7}$, (*b*) $\frac{5x^4 + 7x^2 + 15x + 20}{(x + 2)^5}$,

(*c*) $\frac{4x^5 + 11x^4 + 17x^2 + 35}{(2x^2 + x + 4)^3}$; (*d*) $\frac{4x^4 - 6x^3 + 5x^2 - 8x + 4}{(3x^2 - 4x + 2)^2}$.

3. Solve problem 2(*b*) by the method of undetermined coefficients

CHAPTER XVI

PROBABILITY

At times it is important as well as interesting to be able to compute the probability that an event will happen. A life insurance company requires a larger annual premium on a straight life policy from a man who takes out his policy at the age of thirty-five years than it requires of a man who begins his annual payments at the age of twenty-five years. For a given age, the receipts of the company from a sufficiently large number of insured persons must be adequate for the insurance payments needed.

In order to make an equitable determination of premiums, actuaries have constructed mortality tables which show the expectancy of life for a given group of people. In this chapter will be given the mathematical theory on which such computations are based.

83. Probability. An act which gives an event an opportunity to happen or to fail to happen is called a **trial.** When a coin is tossed there is no reason to assume that one side will turn up rather than the other. We say that the two events, *viz.*, that the coin will land with the head up and that the coin will land with the tail up, are **equally likely** and are also **mutually exclusive,** since the happening of either event makes it impossible for the other to happen.

The probability p that the coin lands with head up is said to be ½ and the probability that it lands with the tail up is also ½.

Definition. *If there are r cases of which h are favorable to a certain event and the remaining r − h cases are unfavorable to it, the probability that the event will happen is, by definition,*

$$p = \frac{h}{r}, \tag{1}$$

and the probability that the event will fail is

$$p' = \frac{r-h}{r}.$$

The above definition is to be applied only when the success and the failure of an event are equally likely. In applying it, it is important that no cases be overlooked.

ILLUSTRATIVE EXAMPLE 1. Let two coins be tossed together. Compute the probability that both will land with head up.

SOLUTION. There are four possibilities or cases.

1. The first coin lands head and the second coin lands tail.
2. The first coin lands tail and the second coin lands head.
3. Both coins land head.
4. Both coins land tail.

Hence the probability for both heads or for both tails is $\frac{1}{4}$, while the probability for one head and one tail is $\frac{1}{2}$.

ILLUSTRATIVE EXAMPLE 2. From a box containing nine white balls and six black balls, five balls are to be drawn at random. What is the probability that there will be three white balls and two black balls?

SOLUTION. The total number of cases is the number of combinations of 15 balls taken 5 at a time; *i.e.*, $r = {}_{15}C_5 = 3{,}003$. The number of cases favorable to 3 white balls is ${}_9C_3 = 84$; the number of cases favorable to 2 black balls is ${}_6C_2 = 15$. Since 3 white balls can be chosen in 84 ways and 2 black balls in 15 ways, the total number of ways in which the two events can happen simultaneously (or in succession) is $84 \cdot 15 = 1{,}260$. Hence

$$p = \frac{1{,}260}{3{,}003} = \frac{60}{143}.$$

ILLUSTRATIVE EXAMPLE 3. If five coins are tossed, what is the probability that they will land three heads and two tails?

SOLUTION. There are two choices for the way in which the first coin lands, two choices for the way in which the second coin lands, and so on for each of the five coins. Hence the number of cases is $r = 2^5 = 32$

The number of favorable cases, since the order of arrangement is immaterial, is equal to the number of ways of arranging five coins with three heads up; *i.e.*, ${}_5C_3 = 10 = {}_5C_2$. Hence the probability required is $\frac{10}{32}$.

Note. The sum of the probabilities that an event will happen and that it will fail is unity since $p + p' = \frac{h}{r} + \frac{r - h}{r} = 1$.

When an event is sure to fail, we have $p = 0$, since the total number of favorable cases is zero; also, $p' = 1$. Similarly, if the event is sure to happen, there are no unfavorable cases and $p = 1$, $p' = 0$. In all other situations the probability p is a positive proper fraction.

EXERCISE LXI

1. What is the probability of tossing four heads at one throw with seven coins? Of tossing three heads?

2. What is the probability of tossing k heads at one throw with n coins? *Ans.* ${}_nC_k/2^n$.

3. An ordinary die has six faces numbered 1, 2, 3, 4, 5, 6, respectively:

(*a*) What is the probability of throwing three with a die?

(*b*) What is the probability of throwing two fives with two dice?

(*c*) What is the probability of throwing a five, a two, and an ace with three dice?

(*d*) What is the probability of throwing a five with one die and at the same time an ace or a two with the other?

4. Find the probability of drawing* a yellow ball from a bag containing six white balls, five black balls, and three yellow balls.

5. If 3 balls are to be drawn from a bag containing 10 balls, what is the probability that a certain ball will be drawn?

6. From a pack of 52 cards one card is drawn. What is the probability that it will be a deuce? A deuce or an ace?

7. If the odds are 5:2 in favor of winning $35, find the player's expectation.

DEFINITION. If p is the probability that a person will receive a sum of k dollars, his **mathematical expectation** is said to be pk.

8. If the probability of losing a game is $\frac{3}{5}$, what is the probability of winning?

9. From a group of seven men and eight women a committee of four is selected by lot. Find the probability that it will consist of two men and two women; of three men and one woman; of four men.

84. Total probability. Compound probability. We denote the probability that an event will occur in a single trial by p. For n trials the *expected number of occurrences* of the event is np. When there are several events which may occur in a single trial, the expected occurrences of each of them in n trials are denoted by np_1, np_2, etc., where p_1, p_2, etc., are the respective probabilities for a single trial.

DEFINITION. The **total probability** P for an occurrence is the ratio of the expected number of occurrences to the number of trials.

The total probability for an occurrence will depend on whether the several events of this occurrence are mutually exclusive, dependent, or independent. These terms will be discussed in the next paragraphs.

It is sometimes convenient to think of an event as being composed of several simpler events. For example, if we are tossing two coins, we may prefer to think of what happens (heads or tails) when each of the coins is tossed separately and speak of the result of tossing two coins together as a **compound event.**

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* All drawings referred to in this chapter are supposed to be random drawings.

Again, if three balls are to be drawn from a bag, they may be drawn simultaneously or one at a time. If drawn one at a time, it is important to know whether or not the first ball is replaced before the second is drawn, etc. The act of drawing three balls simultaneously may be compounded of three separate drawings of one ball each, provided the first and second balls drawn are not replaced in the bag. For such a situation as this we state and prove the following theorem which will be illustrated by examples.

Theorem of compound probability for dependent events. *If the probability of a first event is* p_1, *and if, after this has happened, the probability of a second event is* p_2, *then the probability that both events will happen in the order stated is*

$$P = p_1 p_2.$$

Proof. The cases may be denoted as follows:

h_2 cases favorable to both the first and second events,

h_1 cases favorable to the first but not necessarily to the second event,

k_1 cases unfavorable to the first event.

This means that of the h_1 cases there are h_2 in which both occur. Then, by definition,

$$p_1 = \frac{h_1}{h_1 + k_1},$$

$$p_2 = \frac{h_2}{h_1},$$

$$P = \frac{h_2}{h_1 + k_1}.$$

But the value of P is equal to the product of the two probabilities p_1 and p_2, since

$$p_1 p_2 = \frac{h_1}{h_1 + k_1} \cdot \frac{h_2}{h_1} = \frac{h_2}{h_1 + k_1} = P. \qquad q.e.d.$$

ILLUSTRATIVE EXAMPLE 1. Three balls are to be drawn from a bag containing eight white balls and four black balls. What is the probability that the first two drawn will be white and the last one black?

SOLUTION. The probability that the first ball drawn will be white is $8/12 = 2/3$. If a white ball has actually been drawn, the probability that the second drawing will bring a white ball is $7/11$ Likewise, the probability that the third drawing will bring a black ball is $4/10 = 2/5$. These events are said to be **dependent**, since the result of each drawing depends on the drawings which have preceded it. The (total) probability that the drawing shall bring two white balls and one black ball in the order specified is, by the theorem, the product of the separate probabilities; *viz.*,

$$P = 2/3 \cdot 7/11 \cdot 2/5 = 28/165.$$

The extension of the above theorem to three or more events is obvious.

If the separate events into which a compound event may be decomposed are not dependent, as in the last example, the (total) probability is again the product of the several probabilities as is stated in the next theorem. However, in applying this theorem, the probabilities p_1 and p_2 do not depend on the order of the separate events.

Theorem of compound probability for independent events. *If* p_1 *and* p_2 *are the probabilities of two independent events, the probability that both of these events will happen together at a given trial is*

$$P = p_1p_2,$$

the product of the separate probabilities.

Proof. Assume h_1 cases favorable to one event and k_1 cases unfavorable Then

$$p_1 = \frac{h_1}{h_1 + k_1}.$$

If now there are h_2 cases favorable to the other event and k_2 cases which are unfavorable, we have

$$p_2 = \frac{h_2}{h_2 + k_2}.$$

By hypothesis, the two events are independent, hence the total number of cases to be considered is

$$r = (h_1 + k_1)(h_2 + k_2)$$

In these r cases the two events will happen together in h_1h_2 times. Thus

$$P = \frac{h_1h_2}{r} = \frac{h_1}{h_1 + k_1} \cdot \frac{h_2}{h_2 + k_2} = p_1p_2,$$

where P is the probability that both events may happen together *q.e d.*

The reasoning may be extended to any number m of independent events.

Theorem of probability for mutually exclusive events. *The probability that one or another of a set of mutually exclusive events will happen is the sum of the probabilities of happening for the separate events.*

Proof. We consider only two events and suppose that any trial can result in r cases, of which h_1 are favorable to the first event and h_2 are favorable to the second event. Let p_1 be the probability that the first event will happen and p_2 be the probability for the second event. Then $p_1 = h_1/r$ and $p_2 = h_2/r$.

Since the two events are mutually exclusive, no one of the h_1 cases favorable to the first event is also favorable to the second. Thus $\frac{h_1 + h_2}{r}$ is the number which gives the probability that one or the other of the two events will happen. But $\frac{h_1 + h_2}{r} = \frac{h_1}{r} + \frac{h_2}{r}$ = the sum of p_1 and p_2. *q.e.d.*

EXERCISE LXII

1. In a game in which only one man can win, the probability that Smith will win is 1/2 and that Jones will win is 1/5. What is the probability that either Smith or Jones will win? What is the probability that neither Smith nor Jones will win?

2. The probability that Smith will win in a certain game is 1/3, and that Jones will win in another independent game is 1/5. What is the probability that they will both win?

3. If the probability is 1/3 that the height of a woman selected at random from a group of women is between 5 feet 2 inches and 5 feet 4 inches and 1/4 that it is between 5 feet 4 inches and 5 feet 5 inches, what is the probability that her height is between 5 feet 2 inches and 5 feet 5 inches?

4. Find the probability of drawing two black balls in succession from a bag containing eight black balls and six white balls if the first ball drawn is replaced before the second ball is drawn?

5. Find the probability of drawing two black balls in succession from a bag containing eight black balls and six white balls if the first ball drawn is not replaced before the second drawing.

6. What is the probability that at a given bridge game the next dealer will give 13 cards of the same suit to himself? To one of his opponents?

85. Application to life insurance. When we say that the probability is 7:8 that a forty-year-old man will live to the age of fifty years, we mean that for a certain number of observed cases one-eighth of the men alive at forty died before the expiration of 10 years. This does not mean that if a group of eight forty-year-old men be arbitrarily chosen, one and only one would not live to the age of fifty. The importance of the computed probability depends on the number of cases which have been observed.

Mortality tables which are used by insurance companies for the determination of equitable life insurance premiums are based on a very large number of observed cases. The following items are taken from a mortality table which, starting with 100,000 people living at the age of ten years, gives the numbers living at various ages up to ninety-five years.

Age	Number living
35	81,822
36	81,090
37	80,353
38	79,611
39	78,862
40	78,106

Age	Number living
50	69,804
60	57,917
70	38,569
80	14,474
90	847
95	3

From the above table it is evident that at the age of ten years the probability of living 30 more years is 0.781, while at the age of forty the probability of living 30 more years is 38,569/78,106 = 0.494.

Again, the probability that a person living at the age of sixty will not be living at seventy is 19,348/57,917 = 0.334.

EXERCISE LXIII

1. What is the probability that a man who takes out a 20-year endowment policy at the age of forty will live until the policy matures?

2. At the age of thirty-five *A* begins paying annual premiums on an annuity policy which is to mature at sixty-five What is the probability that he will live to receive 120 monthly payments?

3. What is the probability that a man who is celebrating his ninetieth birthday will live to the age of ninety-five?

SUMMARY

New terms.
  Trial, equally likely, mutually exclusive, compound event, mathematical expectancy
Probability.
  General definition.
  Theorems.
    When events are mutually exclusive.
    When events are independent.
    When events are dependent.

SUPPLEMENTARY EXERCISES

In the following set of exercises the numbers correspond to the exercise numbers in the text.

EXERCISE I

1. Perform the operations indicated, expressing the result in each case as a complex number in Cartesian form.

(*a*) $(2 + i) + (4 - 3i)$;
(*b*) $(7 - 2i) - (-3 + 2i)$;
(*c*) $(8 - i) + (3 - 4i)$;
(*d*) $(4 + 3i) + (-4 + i)$;
(*e*) $(1 + i) - (5 + 2i)$;
(*f*) $(2 - 5i) - (-2 - 5i)$;
(*g*) $(2 + i)(4 - 3i)$;
(*h*) $(5 + i)(5 - i)$;
(*i*) $(3 - i)(2 + 3i)$;
(*j*) $(2 + 3i)(3 - 2i)$;
(*k*) $\frac{4 + 2i}{3 - i}$;
(*l*) $\frac{3 - 3i}{5 - 2i}$;
(*m*) $\frac{7 + 2i}{3 - 4i}$;
(*n*) $\frac{4 + 3i}{4 - 3i}$;
(*o*) $\sqrt{15 - 8i}$;
(*p*) $\sqrt{4 - 3i}$.

2. Construct line segments whose lengths are, respectively, $\sqrt{17}$, $\sqrt{14}$, and $\sqrt{30}$.

EXERCISE II

1. Plot the points in the Argand plane which are associated with the following complex numbers:

$$-1 + i\sqrt{3};\quad 4 + 4i;\quad 2 + 2i;\quad 1 - 3i;\quad -5i;\quad 7i;\quad 4 - i.$$

2. Find the moduli of the numbers in Problem 1.

3. Find the sines and cosines of the amplitudes of the numbers in Problem 1.

4. Plot the points which are associated with the conjugates of the numbers in Problem 1.

5. Express in trigonometric form:

$$3 - 3i\sqrt{3};\quad 4 - 4i;\quad 5i;\quad 1 - i;\quad -1 + i;\quad -10;\quad 2\sqrt{3} - 2i.$$

6. Write in Cartesian form the following complex numbers:

(*a*) $4(\cos 300° + i \sin 300°)$;
(*b*) $3(\cos 120° + i \sin 120°)$;
(*c*) $2(\cos 270° + i \sin 270°)$;
(*d*) $2(\cos 270° - i \sin 270°)$;
(*e*) $6(\cos 45° + i \sin 45°)$;
(*f*) $4(\cos 315° + i \sin 315°)$.

7. Plot the points associated with the complex numbers in Problem 6.

EXERCISE III

1. Add, as indicated, geometrically and arithmetically:

(*a*) $(3 + 4i) + (-4 - 3i)$;
(*b*) $(5 - 2i) + (5 + 2i)$;
(*c*) $3(\cos 120^\circ + i \sin 120^\circ) + (\cos 60^\circ + i \sin 60^\circ)$;
(*d*) $(2 + 7i) + (5i)$.

2. Subtract, as indicated, geometrically and arithmetically:

(*a*) $(8 + 13i) - (2 + 8i)$;
(*b*) $(-3 - i) - (4 - i)$;
(*c*) $2(\cos 240^\circ + i \sin 240^\circ) - 4(\cos 120^\circ + i \sin 120^\circ)$.

EXERCISE IV

1. Construct geometrically the points associated with the products indicated:

(*a*) $(1 - i)(2i)$;
(*b*) $(1 - i)(-2i)$;
(*c*) $(3 + 4i)(3 - 4i)$;
(*d*) $(4 - i)(-4 + i)$.

2. Write the products in Problem 1 in Cartesian form and in trigonometric form.

3. Construct geometrically the points associated with the quotients indicated:

(*a*) $\dfrac{3 - 2i}{2i}$;
(*b*) $\dfrac{-4 - 2i}{-2i}$;
(*c*) $\dfrac{-3 - 4i}{4 - 3i}$;
(*d*) $\dfrac{2 + i}{2 + i}$.

4. Express the quotients in Problem 3 in trigonometric and in Cartesian form.

EXERCISE V

1. Find the roots of the following binomial equations:

(*a*) $x^4 - 5 = 0$;
(*b*) $x^4 + 5 = 0$;
(*c*) $x^3 - 64 = 0$;
(*d*) $x^3 + 64 = 0$;
(*e*) $x^2 = 15 - 8i$;
(*f*) $x^2 = 4 - 3i$.

2. Plot the points associated with the solutions in Problem 1.

EXERCISE VI

1. Find an equation whose roots are $2 + 3i$ and $2 - 3i$.

2. Write equations whose roots are the following:

(*a*) $1, 2, -3$;
(*b*) $4, 2, -6$;
(*c*) $-3, 0, 0, 3$;
(*d*) $1, 3, -1, -3$;
(*e*) $2 + 2i, 2 - 2i, 4$;
(*f*) $(4 - i), (4 + i), 0, -1$.

3. Use equations (7) to find the equations whose roots have the properties indicated:

(*a*) The sum of the three roots of a cubic equation is -2 and the roots are in arithmetic progression with a common difference of 2.

(*b*) The sum of the four roots of a quartic equation is -10, the product is 1,024, and the roots are in geometric progression with ratio equal to -2.

(*c*) One of the roots of $x^3 - 3x^2 + ax + 24 = 0$ is 2; find the value of a.

EXERCISE VII

Find the integral roots, if any, of the following equations:

(*a*) $x^4 + 2x^3 - 4x^2 - 23x - 30 = 0;$
(*b*) $2x^4 - 9x^3 + x^2 + 36x - 36 = 0;$
(*c*) $x^3 - 7x^2 + 3x + 16 = 0;$
(*d*) $2x^3 + 5x^2 - 4x - 3 = 0.$

EXERCISE VIII

1. Multiply the roots of each of the following equations by the number at its right:

(*a*) $x^4 + 2x^3 - 4x^2 - 23x - 30 = 0$ $\quad -2.$
(*b*) $2x^4 - 9x^3 + x^2 + 36x - 36 = 0$ $\quad 2.$
(*c*) $2x^3 + 5x^2 - 4x - 3 = 0$ $\quad -4.$
(*d*) $x^4 + 7x - 5 = 0$ $\quad -1.$

2. By what numbers may the roots of the following equations be multiplied in order that the resulting equation may have its leading coefficient unity and the remaining coefficients integers? Write the transformed equations.

(*a*) $4x^3 + 4x^2 + 5x - 3 = 0,$ (*b*) $16x^3 - 8x + 1 = 0,$
(*c*) $3x^4 - 4x^3 + 2x + 5 = 0,$ (*d*) $5x^3 + 10x + 3 = 0.$

3. Find all of the rational roots of the following equations:

(*a*) $6x^3 - 13x^2 + x + 2 = 0,$
(*b*) $4x^4 - 15x^2 + 6x = 0,$
(*c*) $25x^4 + 25x^3 - 151x^2 - x + 6 = 0;$
(*d*) $3x^4 - x^3 - x^2 - 10x + 8 = 0,$
(*e*) $3x^4 - 7x^3 + 3x - 7 = 0$

EXERCISE IX

From the following equations obtain new equations whose roots are the roots of the given equation diminished in each case by the number in the parentheses.

(*a*) $2x^3 + x^2 + 5x - 8 = 0$ $\quad (1)$
(*b*) $3x^4 - 7x^2 - 2x + 4 = 0$ $\quad (2)$
(*c*) $5x^3 - 2x^2 - 3x + 1 = 0$ $\quad (0\ 1)$
(*d*) $8x^4 + 6x^3 - 5x^2 - 10 = 0$ $\quad (-1).$

EXERCISE X

Locate the irrational roots of the following equations:

(*a*) $2x^3 + 4x^2 - 3x + 4 = 0,$ (*b*) $x^3 + x^2 + 2x - 5 = 0;$
(*c*) $5x^4 + 2x^3 + x^2 - 3x - 8 = 0;$ (*d*) $4x^3 - x^2 - 3x - 3 = 0;$
(*e*) $5x^4 + 8 = 0;$ (*f*) $2x^3 - 7 = 0.$

EXERCISE XI

Find the rational roots, locate the irrational roots, and, when possible, determine the number of complex roots of the following equations:

(*a*) $6x^4 + 19x^3 + 18x^2 + 17x + 20 = 0$;
(*b*) $2x^4 - 9x^3 + 16x - 72 = 0$;
(*c*) $x^3 - 3x^2 + 15x - 26 = 0$;
(*d*) $x^5 - 2x^4 - 2x + 4 = 0$;
(*e*) $x^5 - 5x^3 + 6 = 0$;
(*f*) $4x^3 - 8x^2 + 5x - 1 = 0$.

EXERCISE XII

1. Find all of the real roots of

(*a*) $x^4 - 2x^3 + 4x^2 - 15x + 14 = 0$;
(*b*) $x^3 + 3x - 20 = 0$;
(*c*) $x^4 - 4x - 2 = 0$;
(*d*) $3x^4 + 10x^3 - 9x^2 - 40x - 13 = 0$;
(*e*) $x^4 + 8x^3 + 9x^2 - 8x - 10 = 0$.

2. A sphere of yellow pine 1 foot in diameter floating in water sinks to a depth x given by

$$2x^3 - 3x^2 + 0.6572 = 0.$$

Find the depth to four significant figures.

3. A sphere of ice 1 foot in diameter floating in water sinks to a depth x given by the equation

$$2x^3 - 3x^2 + 0.93 = 0.$$

Find the depth to two significant figures.

4. The width of the strongest beam which can be cut from a log 12 inches in diameter is given by the positive irrational root of the equation

$$x^3 - 144x + 665 = 0.$$

Find the width to two decimal places.

EXERCISE XIII

Perform the indicated additions and subtractions when possible, simplifying radicals.

(*a*) $3\sqrt{24} + 7\sqrt{216} - 5\sqrt{54}$; (*b*) $\sqrt[3]{40{,}000} + 2\sqrt[3]{6{,}655}$;
(*c*) $2\sqrt{4x + 4y} + 5\sqrt{9x + 9y}$; (*d*) $\sqrt{11{,}200} - 5\sqrt{45{,}927}$;
(*e*) $3\sqrt{12x^2y} - 2\sqrt[4]{144x^4y^2}$; (*f*) $4\sqrt{15} + 3\sqrt{540}$.

EXERCISE XIV

Perform the indicated operations, giving the results in simplified form and with no radicals in a denominator.

(a) $4\sqrt{27}\cdot\sqrt{12}$; (b) $4\sqrt{27} \div \sqrt{12}$;

(c) $\dfrac{\sqrt{3} - 4\sqrt{2}}{\sqrt{2} + 4\sqrt{3}}$; (d) $\dfrac{\sqrt{6} - \sqrt{2}}{\sqrt{24} + \sqrt{8}}$;

(e) $(4\sqrt{5} - 1)^2$; (f) $(\sqrt{2} + \sqrt{5})(\sqrt{2} - \sqrt{5})$.

EXERCISE XV

Solve the following equations for x:

(a) $12x + 3\sqrt{3x} - 7 = 0$; (b) $2x + 3 + \sqrt{2x + 5} = 0$;

(c) $\sqrt{2x + 7} - 3 = \dfrac{4}{\sqrt{2x}} + \dfrac{x}{2}$; (d) $\sqrt{2x + 7} = 4 - \sqrt{x + 12}$.

EXERCISE XVI

Find whether the following triples of equations are consistent and, if so, find the common solution.

(a)
$$\begin{aligned} 4x - 6y - 5z &= -1,\\ -x - y + 2z &= 3,\\ 3x + 5y - z &= 0. \end{aligned}$$

(b)
$$\begin{aligned} 2x + y - 3z &= 5,\\ 4x + 3y - z &= 8,\\ 2x + 3y + 7z &= 1. \end{aligned}$$

(c)
$$\begin{aligned} 2x + y - 3z &= 5,\\ 4x + 3y - z &= 8,\\ 4x + 2y + 5z &= -1. \end{aligned}$$

EXERCISE XVII

1. Evaluate the following determinants:

(a) $\begin{vmatrix} -3 & 3 & 5 \\ 1 & -2 & 3 \\ 4 & 2 & -7 \end{vmatrix}$. (b) $\begin{vmatrix} 7 & -2 & 4 \\ -3 & 0 & 3 \\ 4 & 5 & -1 \end{vmatrix}$.

(c) $\begin{vmatrix} 4 & -3 & 2 \\ 1 & 5 & 1 \\ -3 & 2 & 4 \end{vmatrix}$. (d) $\begin{vmatrix} 5 & 4 & 3 \\ -2 & 9 & 8 \\ -7 & 11 & -4 \end{vmatrix}$.

2. Find the value of the following *skew-symmetric* determinants:

(a) $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$. (b) $\begin{vmatrix} 0 & -10 & 7 \\ 10 & 0 & 3 \\ -7 & -3 & 0 \end{vmatrix}$.

(c) $\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$.

EXERCISE XVIII

Classify the following systems of equations geometrically. If they are consistent, find the common points.

(a)
$$\begin{aligned} 3x + 2y + 4z &= -4,\\ 2x - 5y - z &= 20,\\ x + y - z &= 0. \end{aligned}$$

(b)
$$\begin{aligned} 4x - 5y + 6z &= 23,\\ 2x + 7y + 3z &= 40,\\ 9x + y + 8z &= 61. \end{aligned}$$

(c) $$\begin{aligned} \frac{x}{2} + \frac{y}{3} + z &= 1, \\ x - \frac{y}{3} - \frac{z}{2} &= 1, \\ -\frac{x}{3} - y + \frac{z}{2} &= 1. \end{aligned}$$

(d) $$\begin{aligned} \frac{x}{a} + \frac{y}{b} + \frac{z}{c} &= 3, \\ \frac{x}{a} - \frac{y}{b} + \frac{z}{c} &= 1, \\ -\frac{x}{a} + \frac{y}{b} - \frac{z}{c} &= -1. \end{aligned}$$

EXERCISE XIX

1. Compute the areas of the triangles whose vertices are as follows:
(*a*) $(2, -4)$, $(5, 3)$, $(-2, 1)$; (*b*) $(2, -4)$, $(5, 3)$, $(6, -1)$.

2. Write in determinantal form the equations of the lines determined by the points in Problem 1.

3. Does the point $(1, 1)$ lie on a side of either triangle in Problem 1?

4. Which of the following triples of lines are concurrent?

(*a*) $$\begin{aligned} 4x - 3y &= 10, \\ -x + 2y &= 4, \\ 2x + y &= 18. \end{aligned}$$

(*b*) $$\begin{aligned} 3x - 7y &= 0, \\ 2x + y &= -1, \\ x - 3y &= 5. \end{aligned}$$

(*c*) $$\begin{aligned} x + 3y &= 8, \\ 2x - y &= 6, \\ 5x + 2y &= 1. \end{aligned}$$

(*d*) $$\begin{aligned} 4x - 5y &= 7, \\ -x + 2y &= 3, \\ 7x - y &= 2. \end{aligned}$$

EXERCISE XX

1. Draw the graphs of $y = \sec 2x$ and $y = \sec \frac{x}{2}$.

2. Solve the equation $\csc 2x = -2$.

3. Find the remaining functions of x if $\sec 2x = 7/2$.

4. Find positive angles not greater than 360° whose secants are respectively equal to the following:
$\sec 800°$, $\sec 447°$, $\sec(-100°)$, $\csc 77°$, $\csc(-77°)$, $\csc 430°$.

EXERCISES XXI AND XXII

1. Find the derivatives of the following functions:

(*a*) $y = \sin 2x + x^2$; (*b*) $y = \sec 2x - 3x^3$;
(*c*) $y = \csc x^2 - \sin x^2$; (*d*) $y = \text{arc sec } \sqrt{x}$;
(*e*) $y = \cos 3x \csc 2x$; (*f*) $y = \text{arc csc } 4x^2$;
(*g*) $y = \tan 5x \sin 5x$; (*h*) $y = \cos^3 x^3$.

2. Reduce each of the following to a rational expression involving only sines and cosines:

(*a*) $\tan^2 x \csc^2 x - 1$; (*b*) $\dfrac{\tan x + \cot x}{\sec x}$;
(*c*) $(1 + \sin A)(\sec A - \tan A)$; (*d*) $2 \text{ vers } A - \text{vers}^2 A$;
(*e*) $\sec^2 x + \tan^2 x - 5$; (*f*) $\cot x - \sec x \csc x$.

EXERCISE XXIII

1. Prove the following trigonometric identities:

(*a*) $\dfrac{\cot x \cos x}{\cot x + \cos x} = \dfrac{\cot x - \cos x}{\cot x \cos x}$; (*b*) $\dfrac{\sec x + \csc x}{\sec x - \csc x} = \dfrac{1 + \cot x}{1 - \cot x}$;

(*c*) $\cos^2 x + \sin^2 x \sin^2 y + \sin^2 x \cos^2 y = 1$;

(*d*) $(\cos A \cos B + \sin A \sin B \sin C)^2 + (\sin A \cos B - \cos A \sin B \cos C)^2 = 1 - \sin^2 B \sin^2 C$.

2. Solve the following trigonometric equations:

(*a*) $\tan \theta = 2 \sin \theta$; (*b*) $3 \tan^2 \theta - 4 \sin^2 \theta = 1$;
(*c*) $\sin x - \cos x = 5/2$; (*d*) $\sin \theta + \cos \theta = 1$;
(*e*) $\tan \theta + \cot \theta = 2$; (*f*) $2 \sin^2 x = 3 \cos x$.

EXERCISE XXIV

The sides of a triangle are $a = 40$, $b = 32$, $c = 56$.

(*a*) Find the radius of the inscribed circle.
(*b*) Find the radius of the circumscribed circle.
(*c*) Find $\tan \frac{1}{2}A$.
(*d*) Find the distance from B along BA to the inscribed circle.
(*e*) Find the area.

EXERCISE XXV

Given $A(2, -1)$, $B(3, 0)$, $C(-2, 5)$:

(*a*) Find the lengths of the line segments AB and AC.
(*b*) Find the midpoints of the line segments AB and BC.
(*c*) Find the points on AB and AC for which $r = -2/3$.
(*d*) Is the triangle ABC a right triangle?
(*e*) What is the ratio of BP to BC if P is the point $(2, 1)$?

(*f*) Find the equation of the median on the side AB of the triangle ABC; of the altitude on AB.

(*g*) Write the equation of the line through B which is parallel to AC. Put this equation into normal form.

EXERCISES XXVI AND XXVII

1. Make an accurate construction of the ellipses whose equations are:

(*a*) $\dfrac{x^2}{25} + \dfrac{y^2}{81} = 1$; (*b*) $\dfrac{x^2}{81} + \dfrac{y^2}{25} = 1$.

2. Write the coordinates of the foci of the ellipses in the last problem.

3. Write the equations and construct the graphs of the two ellipses for which $a = 5$ and $b = 4$. Locate their foci.

EXERCISES XXIX AND XXX

1. Write the equation of the line which is tangent to the circle $x^2 + y^2 = 25$ at the point $(-3, 4)$.

2. Write the equation of the line which is tangent to the ellipse $\dfrac{x^2}{153} + \dfrac{y^2}{17} = 1$ at the point $P(-3, 4)$.

3. Compute the length of the focal radii to P.

4. Find the equations of the two lines with slope $(-1/12)$ which are tangent to the ellipse in Problem 2. Write the equation of the diameter which is conjugate to the diameter whose slope is $(-1/12)$.

EXERCISE XXXI

1. Construct geometrically the tangent to $\frac{x^2}{9} + \frac{y^2}{36} = 0$ at the point P whose abscissa is $-3/2$ and whose ordinate is positive. Write the equation of the tangent which you constructed.

2. Write the equations of the normal and the two focal radii to the point P in the last problem.

3. Compute the lengths of the two focal radii.

EXERCISE XXXII

1. Write the equations of the tangents to $\frac{x^2}{15} - \frac{y^2}{75} = 1$ at the two points whose abscissa is 8.

2. There are two lines with slope 3 which are tangent to the hyperbola in Problem 1. Write their equations.

3. Find the points of tangency of the lines in the last problem.

EXERCISES XXXIII AND XXXIV

1. Draw the graph of the hyperbola whose equation is $\frac{x^2}{49} - \frac{y^2}{64} = 1$; draw its asymptotes and write their equations.

2. Write the equation of the conjugate hyperbola.

3. Which of these hyperbolas has tangents with slope 1?

4. Find the value of k for which the line $2x - 3y + k = 0$ is tangent to one of the hyperbolas in Problems 1 and 2. To which of the hyperbolas is it tangent?

5. Write the equation of the diameter which is conjugate to the diameter whose equation is $2x - 3y = 0$.

EXERCISE XXXV

Verify the following statements for the hyperbola whose equation is $\frac{x^2}{36} - \frac{y^2}{9} = 1$.

(*a*) The eccentricity is $\frac{1}{3}\sqrt{11}$.

(*b*) The equation of the tangent at $M(10, -4)$ is $5x + 8y - 18 = 0$

(*c*) The coordinates of the foci C and C' are $(2\sqrt{11}, 0)$ and $(-2\sqrt{11}, 0)$, respectively.

(*d*) The equation of the normal at M is $8x - 5y - 100 = 0$.

(*e*) The slopes of the focal radii are $\frac{5 \pm \sqrt{11}}{7}$.

(*f*) The angle between the focal radii is arc tan $\frac{2\sqrt{11}}{5}$.

(*g*) No line with slope less than $\frac{1}{2}$ can be tangent to the hyperbola.

EXERCISE XXXVI

Write the equation of the hyperbola with center at $(0, -3)$, transverse diameter = 4, conjugate diameter = 5, and with the conjugate axis horizontal.

(*a*) Compute the length of a latus rectum.
(*b*) Locate the foci.
(*c*) Write the equation of the conjugate hyperbola.
(*d*) Compute the length of a latus rectum of the conjugate hyperbola.
(*e*) Compute the two eccentricities.
(*f*) Write the equation of the tangent at the point whose ordinate is zero and whose abscissa is negative.
(*g*) Find the area of the triangle whose sides are the x-axis, the y-axis, and the tangent line in (*f*).

EXERCISE XXXVII

1. Find the equations of the tangents to the following parabolas at the points indicated. Draw the graphs. Locate the foci and write the equations of the directrices.

(*a*) $y^2 = -3x$, $(-3, -3)$;
(*b*) $y^2 = 5x$, $(1, \sqrt{5})$;
(*c*) $3y = 4x^2$, $(-1, \frac{4}{3})$;
(*d*) $5y = 8x^2$, $(-2, \frac{32}{5})$.

2. Compute the length of the latus rectum of each of the parabolas in the last problem.

3. Find the coordinates of the focus of the parabola whose equation is $6(x + 1) = (y - 5)^2$. Write the equation of the directrix.

4. Write the equation of the directrix and the coordinates of the focus of the parabola whose equation is $6(y - 5) = (x + 1)^2$.

5. Show that the line $x - 2y + 11 = 0$ is tangent to the parabola $6(y - 5) = (x + 1)^2$. Find the point of tangency.

EXERCISE XXXVIII

Sketch the following curves, showing foci and directrices. Write the coordinates of the foci and of the centers. Also write the equations of the directrices.

(*a*) $x^2 - y^2 = -4$;
(*b*) $3x^2 + 6x + 5y^2 = 40$;
(*c*) $2x^2 - y^2 = 4$;
(*d*) $5x^2 - 15x - 8y^2 = 36$;
(*e*) $3x^2 - y^2 = 4$;
(*f*) $2x^2 + 8y^2 - 20y = 25$.

EXERCISE XXXIX

Reduce each of the following equations to one of the type forms in equations (10) on page 101. Interpret.

(*a*) $x^2 + 6x + y^2 - 8y + 15 = 0$;
(*b*) $x^2 + 6x + y^2 - 8y + 25 = 0$;
(*c*) $x^2 + 6x + y^2 - 8y + 35 = 0$;
(*d*) $x^2 - 4y^2 = 0$;
(*e*) $x^2 - 4y^2 = 8$;
(*f*) $x^2 - 4y^2 = -8$;
(*g*) $x^2 + 4y^2 = 0$;
(*h*) $x^2 + 4y^2 = -8$.

EXERCISE XL

1. Use the method of Fig. 44 to construct 12 points of an ellipse whose eccentricity is $\frac{3}{4}$.

2. Use the method of Fig. 45 to construct 12 points of a hyperbola whose eccentricity is 3.

3. Use the method of Fig. 46 to construct 12 points of an ellipse whose semi-axes have the respective values $a = 4$, $b = 8$.

4. Use the method of Fig. 47 to construct 12 points of a hyperbola whose transverse semi-diameter and conjugate semi-diameter have the values $a = 4$, $b = 8$.

5. Construct in two ways a tangent to the ellipse in Problem 3.

EXERCISE XLI

1. When the frame of reference is translated 4 units to the left and 3 units up, what are the new coordinates of the points whose coordinates when referred to the original reference frame were as follows:

$$(1, 1), \qquad (3, -5), \qquad (0, 4), \qquad (2, 6), \qquad (-1, 3), \qquad (-4, 3)?$$

2. When the frame of reference is rotated through 120°, what are the new coordinates of the points whose coordinates when referred to the original frame were as follows:

$$(1, 1), \qquad (2, 2), \qquad (3, 3), \qquad (-3, -3), \qquad (5, -2), \qquad (0, 0)?$$

3. Write the equations of rotation through an angle of $-45°$.

EXERCISE XLII

Find a favorable angle of rotation for each of the following. Construct the graphs.

(a) $4x^2 - 5xy + y^2 = 9$;
(b) $4x^2 - 5xy - y^2 = 9$;
(c) $4x^2 - 4xy + y^2 = 9$;
(d) $4x^2 - 4xy - 9y^2 = 9$;
(e) $8x^2 + 14xy + 5y^2 = 0$.

EXERCISE XLIII

1. Find the type of each of the following conics and determine whether it is degenerate or non-degenerate.

(a) $2x^2 + 7xy - 3y^2 + 7x + 3y = 0$;
(b) $3x^2 + 8xy + 3y^2 - 5x - 4y = 0$;
(c) $3x^2 + xy + 4y^2 - 2x + 3y = 0$;
(d) $x^2 - 5xy = 0$.

2. Show that the following conics are non-degenerate. Sketch the graphs.

(a) $3x^2 - 4xy - 2x + 4y^2 + 6y - 5 = 0$;
(b) $2x^2 - 8xy + 6x + 5y^2 - 2y + 8 = 0$.

EXERCISES XLIV AND XLV

Construct the graphs of the following:

(a) $r = 3$;

(b) $r = 4 \cos \frac{\theta}{2}$;

(c) $\theta = 40°$;

(d) $r = 4 \cos 2\theta$;

(e) $r = \sin \theta$;

(f) $r = 2 \sin \theta$;

(g) $r = \sin \frac{\theta}{2}$;

(h) $r = 2 \sin \frac{\theta}{2}$

EXERCISE XLVIII

1. Make the graphs of the curves whose equations follow and find the Cartesian equation for each curve.

(a) $x = t - 1,\quad y = 2t^2 + 5$;

(b) $x = m^2 - 4,\quad y = m^3 - 4m$;

(c) $x = \dfrac{1}{t - 1},\quad y = t + 5$;

(d) $x = \dfrac{2}{t + 3},\quad y = \dfrac{1}{t^2}$.

2. Obtain the Cartesian equation of the *folium of Descartes* whose parametric representation may be written

$$x = \frac{3am}{1 + m^3}, \quad y = \frac{3am^2}{1 + m^3}$$

Ans. $x^3 + y^3 - 3axy = 0$.

EXERCISE XLIX

1. Write the equation of the pencil of lines through the point $A(-1, 3)$ Find the line of this pencil which passes through the origin

2. Write the equation of the pencil of lines through the intersection of $3x - y + 4 = 0$ and $2x + 3y - 1 = 0$

3. Given two circles with centers at (0, 0) and (4, −3) and respective radii equal to 1 and 2.

(a) Write the equation of the pencil of circles determined by them

(b) Write the equation of the line of centers and of the radical axis

(c) Compute the lengths of the tangents to the given circles from (2, 5)

(d) Write the equation of the circle of the pencil which passes through (4, 0)

EXERCISE L

1. Set up the discriminant of the pencil of conics determined by

S_1: $3x^2 + 8xy - 14y^2 + 3x + 20y = 0$,
S_2: $3x^2 - 2xy + 11y^2 + 3x - 35y = 0$.

Find the three degenerate conics of the pencil.

2. Find the equation of the conic which passes through the following points:

$$(1, 2), \qquad (-3, 3), \qquad (2, -3), \qquad (-1, 0), \qquad (0, 4)$$

EXERCISES LI AND LII

1. Insert an arithmetic mean and a geometric mean between 7 and 28.

2. The seventh term of an A P is 8 and the twelfth term is -40; find the common difference, the thirteenth term, and the first term

3. The seventh term of a G P is 8 and the twelfth term is -40 Find the ratio, the thirteenth term, and the first term

4. *A* writes five letters to five different friends, asking each of them to vote for *B* for president and to write five similar letters If the letters are numbered serially, *A*'s letters being numbered "one," the letters written by *A*'s five friends being numbered "two," and no letters written with numbers greater than "eight," and if 75 per cent of the persons reached in this way vote for *B*, how many votes will that be, assuming that the chain is not broken until all of the letters numbered "eight" are sent?

EXERCISES LIII AND LIV

1. Find the limit of the sum of an infinite number of terms of the following infinite series.

$$(a)\ 10 + \frac{10}{9} + \frac{10}{9^2} + \qquad ,$$

$$(b)\ 6 - 3 + \frac{3}{2} - \frac{3}{4} + \qquad ,$$

$$(c)\ 1 + x + x^2 + \qquad \left(x = \frac{1}{3}\right)$$

2. Find the limiting value of each of the following non-terminating decimals.

(*a*) 0.714825714825 ,
(*b*) 4.2727 ;
(*c*) 2.8181
(*d*) 1.769230769230

3. Insert one harmonic mean between $\frac{1}{8}$ and 8

4. Find the sixth term of the H P $-\frac{1}{3}, -1,$

EXERCISES LVI AND LVII

1. Write the following expansions

$$(a)\ (a - 2)^5; \qquad (b)\ (2x + y)^6; \qquad (c)\ (a^2 - b^2)^4; \qquad (d)\ (x - y + 1)^4$$

2. Write the eighth term of the expansion of $(4x - 3y)^9$; of $(a + 4b)^{11}$

3. Use the binomial theorem more than once to obtain the expansion of $(a + b + c + d)^4$.

4. How many terms of the expansion of $(32 + 3)^{1/5} = 2(1 + 3/32)^{1/5}$ are necessary in order to obtain $\sqrt[5]{35}$ correct to four decimal places?

EXERCISES LVIII AND LIX

1. How many permutations can be made with the letters of the word "permutations"?

2. How many combinations can be made with the letters of the word "combinations."

3. How many signals can be given with four flags of different colors by flying them from four permanent flagpoles?

4. In how many ways may six beads of different colors be strung on a closed circular wire?

5. In how many ways can 4 red books, 6 green books, and 4 blue books be placed on a shelf if books of the same color stand together?

EXERCISE LX

Decompose the following fractions into simple partial fractions

(*a*) $\dfrac{3x^2 - 17x + 17}{(x - 2)^2(x + 3)}$; (*b*) $\dfrac{13x^2 - 10x + 15}{(2x^2 - 3x + 3)(x^2 + 4)}$;

(*c*) $\dfrac{14x^3 - 10x^2 + 7x - 1}{2x^3(x - 1)}$, (*d*) $\dfrac{x}{x^2 - 4}$,

(*e*) $\dfrac{3x^3 - 5x^2 + 7x + 4}{(x^2 + 1)^2}$, (*f*) $\dfrac{x^3 + 5}{(x^2 - 3)^3}$

EXERCISES LXI AND LXII

1. If four cards are drawn from an ordinary pack, what is the probability that they are all of the same suit?

2. What is the probability that the four cards in Problem 1 are all aces?

3. One box contains five white, four black, and three red balls; a second box contains four white and six black balls. From one of the boxes three balls are drawn. What is the probability that they are all black?

4. In a group of 18 persons, 15 favor a proposition, and 3 oppose it. If 7 people are chosen at random from the group, what is the probability that just 2 of them oppose the proposition?

5. Two sisters are aged respectively thirty-five and thirty-eight years. What is the probability that just one of them will be alive in 25 years?

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